ON THE RELATION BETWEEN GALOIS GROUPS AND MOTIVIC GALOIS GROUPS

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Abstract. Let k be a subfield of $\mathbb{C}$, and let $\overline{k}$ be its algebraic closure in $\mathbb{C}$. We establish a short exact sequence relating the motivic Galois groups of k and of $\overline{k}$ with the absolute Galois group $\mathrm{Gal}(\overline{k}|k)$, in the framework of Nori's category of mixed motives with coefficients in any field. In the case where the coefficient field has positive characteristic, we show that the motivic Galois group of k is isomorphic to $\mathrm{Gal}(\overline{k}|k)$. In the case of rational coefficients, we show that the full subcategory of Nori's category of mixed motives over k whose objects are semisimple is equivalent to André's category of pure motives over k with respect to motivated correspondences. As a consequence, the motivic Galois groups for André's motives are the maximal reductive quotients of the Galois group that one obtains from Nori's categories, and we obtain a similar short exact sequence in the framework of André's motives.

Contents

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Introduction and overview

Let k be a subfield of $\mathbb{C}$, and let $\overline{k}$ be the algebraic closure of k in $\mathbb{C}$. The absolute Galois group $\mathrm{Gal}(\overline{k}|k)$ of k is, from Grothendieck's point of view, the group of automorphisms of the functor

$$\omega : \{\text{Finite étale } k\text{-schemes}\} \longrightarrow \{\text{Finite sets}\}$$

sending an étale k-scheme X to the finite set of its complex valued points $X(\mathbb{C}) = X(\overline{k})$. The group of automorphisms of ω has the structure of a profinite group, and the category of finite étale k-schemes can be recovered as the category of finite sets with a continuous action of the absolute Galois group. In place of the set valued functor ω we can also consider the vector space valued functor

$$\omega_{\mathbb{Q}} : \{\text{Finite étale } k\text{-schemes}\} \longrightarrow \{\text{Finite dimensional rational vector spaces}\}$$

sending an étale scheme X to the vector space $H^0(X(\mathbb{C}), \mathbb{Q})$, which is just the rational vector space with basis $\omega(X) = X(\mathbb{C})$. The group of automorphisms of $\omega_{\mathbb{Q}}$ is still the absolute Galois group of k, but now viewed as an affine group scheme over $\mathbb{Q}$. To give an R-rational point g of the group scheme $\mathrm{Gal}(k) := \mathrm{Aut}(\omega_{\mathbb{Q}})$ for some $\mathbb{Q}$-algebra R is to give for each finite étale k-scheme X an R-linear automorphism g_X of $H^0(X(\mathbb{C}), R)$, and these automorphisms g_X are required to be compatible with morphisms between étale k-schemes.

Much of Grothendieck's interest in tannakian formalism stems from the idea that an analogous mechanism should work in the framework of motives. Motives over k form an abelian, $\mathbb{Q}$-linear category, and there is a faithful, exact functor

$$\mathrm{R_B} : \{\text{Motives over } k\} \longrightarrow \{\text{Finite dimensional rational vector spaces}\}$$

sending a motive to its Betti realisation. The group of automorphisms of $\mathrm{R_B}$ has the structure of an affine group scheme over $\mathbb{Q}$, and is called *Motivic Galois group* or also motivic fundamental group of k. We denote it by $G_{\mathrm{mot}}(k)$. The the category of motives can be recovered as the category of finite dimensional representations of this group scheme. In other words, the category of motives over k is a neutral tannakian category, and Betti realisation is a fibre functor. To give an R-rational point g of the group scheme $G_{\mathrm{mot}}(k) := \mathrm{Aut}^{\otimes}(\mathrm{R_B})$ for some $\mathbb{Q}$-algebra R is to give for each motive M an R-linear automorphism g_M of $\mathrm{R_B}(M) \otimes R$, and these automorphisms g_M are required to be compatible with morphisms and tensor products in the category of motives.

Even without knowing what the category of motives over k concretely is, we obtain from this picture a relation between the usual absolute Galois group of k and the motivic Galois group of k. Indeed, every finite étale k-scheme X yields a motive $H^0(X)$ over k. Motives arising this way are called *Artin motives*. The Betti realisation of $H^0(X)$ is the vector space $\omega_{\mathbb{Q}}(X) = H^0(X(\mathbb{C}), \mathbb{Q})$. We obtain from this a morphism of group schemes

$$G_{\mathrm{mot}}(k) \xrightarrow{\ p\ } \mathrm{Gal}(\overline{k}|k)$$

sending an automorphism g of $\mathrm{R_B}$ to the automorphism $p(g)$ of $\omega_{\mathbb{Q}}$ given by $p(g)_X = g_{H^0(X)}$. The map p was studied by Deligne and Milne in [DM82] in the framework of motives with respect to absolute Hodge cycles. It is shown in Proposition 6.23 of [DM82] that in this context p is surjective, and that its kernel is the connected component of the identity of the group scheme

$\mathrm{Aut}^{\otimes}(\mathrm{R_B})$. The reason for this is essentially that Mumford-Tate groups are connected. This connected component should be the motivic fundamental group of $\overline{k}$, the algebraic closure of k in $\mathbb{C}$, according to the conjectural statement 6.3 in Serre's [Se94]. More precisely, there is a canonical, faithful and exact functor that sends motives over k to motives over $\overline{k}$. This functor is compatible with Betti realisation, hence induces a morphism of group schemes

$$G_{\mathrm{mot}}(\overline{k}) \xrightarrow{\ i\ } G_{\mathrm{mot}}(k)$$

which is injective, and whose image is expected to be the connected component of the unit in $G_{\mathrm{mot}}(k)$, hence the kernel of p.

Since the work of Deligne and Milne who deal with motives for absolute Hodge cycles, and the work of Serre who hacks his way into the paradise of motives by assuming the Hodge conjecture, two new tannakian categories of motives have seen the light of day. The first one is a category of pure motives constructed by André in [And96], and the second one is a category of mixed motives due to Nori [N]. Assuming the Hodge conjecture, the semisimple categories are all equivalent, and Nori's stands apart.

Our aim is to clarify the relation between Galois groups and motivic Galois groups in the context of André's and Nori's categories of motives. Our first result is the following theorem (Theorem 10.7 in the text), which can be interpreted as Galois descent for motives:

Theorem 1. *Let k be a subfield of $\mathbb{C}$ and let $\overline{k}$ be the algebraic closure of k in $\mathbb{C}$. Let $G_{\mathrm{mot}}(k)$ and $G_{\mathrm{mot}}(\overline{k})$ be the tannakian Galois groups of Nori's $\mathbb{Q}$-linear tannakian categories of mixed motives over k and over $\overline{k}$ respectively. The sequence of group schemes*

$$1 \longrightarrow G_{\mathrm{mot}}(\overline{k}) \xrightarrow{\ i\ } G_{\mathrm{mot}}(k) \xrightarrow{\ p\ } \mathrm{Gal}(\overline{k}|k) \longrightarrow 1$$

over $\mathbb{Q}$ is exact. In this sequence, the morphism i is the one induced by base change from motives over k to motives over $\overline{k}$, and the morphism p is the one induced by the inclusion of the subcategory of Artin motives over k into the category of all motives over k.

It would follow from a weak form of the Hodge conjecture that the group scheme $G_{\mathrm{mot}}(\overline{k})$ over $\mathbb{Q}$ is connected, so that the group of components of $G_{\mathrm{mot}}(k)$ is isomorphic to the absolute Galois group of k, seen as a constant group scheme over $\mathbb{Q}$. We will only show a much weaker statement, which is the following: The group scheme $G_{\mathrm{mot}}(k)$ has a canonical flat model over $\mathbb{Z}$, and the component group of this model is $\mathrm{Gal}(\overline{k}|k)$, seen as a constant group scheme over $\mathbb{Z}$. It means, in essence, that to give a motive over $\overline{k}$ with integer coefficients is the same as to give a motive over $\overline{k}$ with rational coefficients together with a lattice in its Betti realisation. This also amounts to:

Theorem 2. *Let k be a subfield of $\mathbb{C}$, let ℓ be a prime number and let $\mathbb{F}_\ell$ be the finite field with ℓ elements. Denote by $\mathbf{M}(k)_{\mathbb{Z}}$ and $\mathbf{M}(k)_{\mathbb{F}_\ell}$ Nori's categories of mixed motives over k with integer coefficients, respectively with coefficients in $\mathbb{F}_\ell$. The following categories are equivalent:*

(1) *The category of finite dimensional, continuous $\mathbb{F}_\ell$-linear representations of* $\mathrm{Gal}(\overline{k}|k)$*.*
(2) *The category $\mathbf{M}(k)_{\mathbb{F}_\ell}$.*
(3) *The full subcategory of $\mathbf{M}(k)_{\mathbb{Z}}$ consisting of those objects which are annihilated by ℓ.*

Theorem 2 shows us, that motives with coefficients in a field of positive characteristic are not so interesting, in the sense that they are not more and not less than Galois representations, and also shows that motives with integer coefficients can be understood in terms of Galois modules and motives with rational coefficients. Mixed motives with rational coefficients form a tanakian category, whose relation with André's semisimple motives is explained by the following theorem. This theorem was already stated by Arapura in [Ara13] in a somewhat different setup of which I don't know whether it is well related to ours.

Theorem 3. *Let k be a subfield of $\mathbb{C}$. Let $\mathbf{M}(k)$ be Nori's $\mathbb{Q}$-linear tannakian category of mixed motives over k, and let $\mathbf{A}(k)$ be André's $\mathbb{Q}$-linear tannakian category of pure motives with respect to motivated correspondences over k. There is a canonical functor*

$$\mathbf{A}(k) \longrightarrow \mathbf{M}(k)$$

which sends the motive in $\mathbf{A}(k)$ of a smooth and proper variety to the motive of the same variety in $\mathbf{M}(k)$. This functor induces an equivalence between André's category $\mathbf{A}(k)$ and the full subcategory of Nori's category $\mathbf{M}(k)$ whose objects are the semisimple objects.

It follows from this theorem that the tannakian Galois group of André's category $\mathbf{A}(k)$ is the maximal reductive quotient $G^{\mathrm{red}}_{\mathrm{mot}}(k)$ of $G_{\mathrm{mot}}(k)$. In particular, these two group schemes have the same component group, and hence Theorem 1 holds analogously for André's motives. The essential step in the proof of Theorem 2 is to show that the functor $\mathbf{A}(k) \longrightarrow \mathbf{M}(k)$, which is not hard to construct, is full. In order to prove that $\mathbf{A}(k) \longrightarrow \mathbf{M}(k)$ is full, we will construct a functor in the opposite direction, of which one should think as associating to a mixed motive the corresponding weight graded semisimple motive, so that the composition $\mathbf{A}(k) \longrightarrow \mathbf{M}(k) \longrightarrow \mathbf{A}(k)$ is an equivalence. The technical tool we use to construct such a functor are the weight complexes introduced by Gillet and Soulé.

The paper is organised as follows: In section 1 we rehearse and give some complements to Nori's tannakian formalism, and in section 2 we introduce Nori's category of mixed motives. Our construction of this category is marginally different from the construction given by Nori in [N] or [HM16] – it suits us best to follow an idea of Ayoub's, as explained in the introductory paragraph to section 2.

Sections 3, 4, 5 and 6, a substantial proportion of the paper, deal with important constructions in Nori's category. This is dry work - the reason for including this material here is primarily that no comprehensive treatment is yet available. The forthcoming book [HM16] should remedy this. In section 7 we introduce motivic fundamental groups.

In section 8 we introduce the weight filtration on $\mathbf{M}(k)$ and prove Theorem 3. In Section 9 we study motives with finite coefficients and prove Theorem 2. The proof of Theorem 1 is finally given in section 10, which is about Galois descent.

1. Complements to Nori's reconstruction theorem

In this section I introduce a categorical construction due to Nori. It attaches to a diagram of modules an abelian category which contains the diagram in a suitable sense, and is universal for that property. To formalise the idea of a diagram of modules, we shall use the language of quivers and quiver representations in a nonstandard way. Although for our applications we only need to work with modules over typical coefficient rings $\mathbb{Z}, \mathbb{Q}, \mathbb{Z}_\ell, \ldots$, we work over a general commutative coherent ring. It costs us no additional effort.

- 1.1. We fix for this section a commutative coherent ring R, and denote by $\mathbf{Modf}_R$ the category of finitely presented R-modules. This is an abelian, monoidal closed category. By an *R-linear* category we mean a category that is tensored over $\mathbf{Modf}_R$. Unless mentioned otherwise, we call objects of $\mathbf{Modf}_R$ just modules, and algebra objects in $\mathbf{Modf}_R$ just algebras.

1.1. **Quivers and quiver representations.**

Definition 1.2. A *quiver* is a tuple $Q = \big(\mathrm{Obj}(Q),\, \mathrm{Mor}(Q),\, i, s, t\,\big)$ where $\mathrm{Obj}(Q)$ and $\mathrm{Mor}(Q)$ are classes, called *objects* and *morphisms* of Q, and where i, s and t are functions

$$\begin{tikzcd} \mathrm{Obj}(Q) \arrow[r, "i"] & \mathrm{Mor}(Q) \arrow[l, bend right, "s"'] \arrow[l, bend left, "t"] \end{tikzcd}$$

called *identity*, *source* and *target*. They have to satisfy $t \circ i = s \circ i = \mathrm{id}$, and given two objects p and q of Q, we require that

$$\mathrm{Mor}(p,q) := \{ f \in \mathrm{Mor}(Q) \mid s(f) = p \text{ and } t(f) = q \}$$

is a set. A *morphism* of quivers $\rho : Q \longrightarrow Q'$ consists of a pair of functions between objects and between morphisms which are compatible with the identity, source and target functions in the obvious way. We say that a quiver is *small* if its objects form a set, and that it is *finite* if it has only finitely many objects and morphisms.

- 1.3. Every category can be viewed as a quiver by forgetting the composition law of morphisms. I will without further comment transport terminology for categories and functors to the setting of quivers and quiver morphisms. One can make out of a small quiver Q a category $\langle Q\rangle$ by taking as morphisms words with letters from the morphism set and specifying the role of the identities. There is then a natural morphism of quivers $Q \longrightarrow \langle Q\rangle$, which, given a category $\mathbf{C}$, induces a bijection between the class of quiver morphisms $Q \longrightarrow \mathbf{C}$ and the class of functors $\langle Q\rangle \longrightarrow \mathbf{C}$. Given a quiver Q, we can form the quiver Q^{op} which has the same objects and morphisms as Q, and whose source and target functions are those from Q but with their roles interchanged.

Definition 1.4. A *representation* of a quiver Q in a category $\mathbf{C}$ is a morphism of quivers ρ from Q to $\mathbf{C}$. A morphism of quiver representations $(Q \xrightarrow{\rho} \mathbf{C}) \longrightarrow (Q' \xrightarrow{\rho'} \mathbf{C})$ consists of a quiver morphism $\varphi : Q \longrightarrow Q'$ and an isomorphism (a natural transform) of quiver morphisms $\rho' \circ \psi \cong \rho$.

- 1.5. Let Q be a finite quiver, and let $\rho : Q \longrightarrow \mathbf{C}$ be a quiver representation of Q in a monoidal closed abelian category $\mathbf{C}$. The endomorphism ring $\mathrm{End}(\rho)$ is the algebra object in $\mathbf{C}$ given by

$$\mathrm{End}(\rho) = \mathrm{equaliser}\Big(\prod_{q \in Q} \mathrm{End}(\rho(q)) \rightrightarrows \prod_{p \to q} \mathrm{Hom}(\rho(p), \rho(q)) \Big) \tag{1}$$

where $\mathrm{End}(\rho(q))$ and $\mathrm{Hom}(\rho(p),\rho(q))$ are the internal homomorphism objects in $\mathbf{C}$. In the case where $\mathbf{C}$ is the category $\mathbf{Modf}_R$ of finitely presented R-modules, $\mathrm{End}(\rho)$ is the R-algebra consisting of tuples $(e_q)_{q\in Q}$ of R-linear endomorhpisms $e_q : \rho(q) \longrightarrow \rho(q)$ such that the squares

$$\begin{array}{ccc} \rho(p) & \xrightarrow{\rho(f)} & \rho(q) \\ {\scriptstyle e_p}\downarrow & & \downarrow{\scriptstyle e_q} \\ \rho(p) & \xrightarrow{\rho(f)} & \rho(q) \end{array}$$

commute for all morphisms $f : p \longrightarrow q$ in Q. This algebra is finitely presented as an R-module[1].

Definition 1.6. Let Q be a quiver, and let $\rho : Q \longrightarrow \mathbf{Modf}_R$ be a quiver representation. We call *linear hull* of (Q,ρ), and denote by $\langle Q,\rho\rangle$, the following category.

- Objects of $\langle Q,\rho\rangle$ are triples (M,F,α) where M is a module, F is a finite subquiver of Q, and α is an R-linear action of the algebra $\mathrm{End}(\rho|_F)$ on M.
- Morphisms $(M_1,F_1,\alpha_1) \longrightarrow (M_2,F_2,\alpha_2)$ in $\langle Q,\rho\rangle$ are linear maps $M_1 \longrightarrow M_2$ with the property that there exists a finite subquiver F_3 of Q containing F_1 and F_2, such that f is $\mathrm{End}(\rho|_{F_3})$-linear. The action of $\mathrm{End}(\rho|_{F_3})$ on M_i is obtained via α_i and the restriction $\mathrm{End}(\rho|_{F_3}) \longrightarrow \mathrm{End}(\rho|_{F_i})$.
- Composition of morphisms in $\langle Q,\rho\rangle$ is composition of linear maps.

- 1.7. Given a quiver representation $\rho : Q \longrightarrow \mathbf{Modf}_R$, its linear hull $\langle Q,\rho\rangle$ is an abelian R-linear category. We can understand it as a concrete instance of the 2-colimit of small categories

$$\langle Q,\rho\rangle = \operatorname*{colim}_{F\subseteq Q} \{\mathrm{End}(\rho|_F)\text{-module objects in } \mathbf{Modf}_R\} \tag{2}$$

and if Q is small, then we can define the proalgebra (that means: proobject in the category of R-algebras) $E := \lim_F \mathrm{End}(\rho|_F)$ and identify $\langle Q,\rho\rangle$ with the category of continuous E-module objects in $\mathbf{Modf}_R$. The forgetful functor $\langle Q,\rho\rangle \longrightarrow \mathbf{Modf}_R$ sending a triple (M,F,α) to the R-module M is exact and faithful. Moreover, the linear hull comes equipped with a representation

$$\widetilde{\rho} : Q \longrightarrow \langle Q,\rho\rangle$$

given by $\widetilde{\rho}(q) = (\rho(q),\{q\},\mathrm{id})$. We call this representation the *canonical lift* of ρ, since its composition with the forgetful functor gives back the original representation ρ.

1.2. **Functoriality.**

- 1.8. An important feature of linear hulls of quiver representations is that they are functorial in the following sense: Given a morphism of quiver representations, that is, a triangle of quiver

[1] We may recognise (1) as part of a certain Hochschild simplicial complex. In particular, if Q has only one object, we recognise a part of the Hochschild complex of the free R-algebra generated by the morphisms of Q acting on the bimodule $\mathrm{End}(\rho(q))$. The Hochschild cohomology vanishes from H^2 on, and the first Hochschild cohomology group, whose elements have the interpretation of derivations modulo inner derivations, is the coequaliser of (1).

morphisms together with a natural transform

$$\begin{tikzcd} Q \arrow[rr, "\varphi"] \arrow[dr, "\rho"'] & & Q' \arrow[dl, "\rho'"] \\ & \mathbf{Modf}_R & \end{tikzcd} \qquad \begin{array}{rl} s: & \rho' \circ \varphi \xrightarrow{\cong} \rho \\ s_q: & \rho'(\varphi(q)) \xrightarrow{\cong} \rho(q) \end{array}$$

we obtain a functor $\Phi : \langle Q, \rho\rangle \longrightarrow \langle Q', \rho'\rangle$ by setting $\Phi(M, F, \alpha) = (M, \varphi(F), \alpha \circ \sigma)$, where $\varphi(F)$ is the image of the finite subquiver $F \subseteq Q$ in Q' under φ, and σ the morphism of algebras $\mathrm{End}(\rho'|_{\varphi(F)}) \longrightarrow \mathrm{End}(\rho|_F)$ obtained from s. In terms of 1.5, the morphism σ sends the tuple $(e_{q'})_{q'\in\varphi(F)}$ to the tuple $(s_q \circ e_{\varphi(q)} \circ s_q^{-1})_{q\in F}$. We notice that the functor Φ is faithful and exact, and that it commutes with the forgetful functors and up to natural isomorphisms with the canonical lifts.

- 1.9. The induced functor Φ in the previous paragraph depends naturally on the morphism of quiver representations (φ, s) in the following sense. Let $\rho : Q \longrightarrow \mathbf{Modf}_R$ and $\rho' : Q' \longrightarrow \mathbf{Modf}_R$ be quiver representations, and let

$$\begin{tikzcd} Q \arrow[rr, shift left, "\varphi"] \arrow[rr, shift right, "\psi"'] \arrow[dr, "\rho"'] & & Q' \arrow[dl, "\rho'"] \\ & \mathbf{Modf}_R & \end{tikzcd} \qquad \begin{array}{l} s : \rho' \circ \varphi \longrightarrow \rho \\ t : \rho' \circ \psi \longrightarrow \rho \end{array}$$

be two morphisms of quiver representations. Denote by Φ and Ψ the induced functors between linear hulls $\langle Q, \rho\rangle \longrightarrow \langle Q', \rho'\rangle$. We call *2-morphism* from (φ, s) to (ψ, t) every natural transform $\eta : \widetilde{\rho}' \circ \varphi \longrightarrow \widetilde{\rho}' \circ \psi$ with the property that for every $q \in Q$ the diagram of R-modules

$$\begin{tikzcd} \rho'(\varphi(q)) \arrow[r, "\eta_q"] \arrow[d, "s_q"] & \rho'(\psi(q)) \arrow[d, "t_q"] \\ \rho(q) \arrow[r, equal] & \rho(q) \end{tikzcd} \tag{3}$$

commutes. Such a 2-morphism η indeed induces a morphism of functors $E : \Phi \longrightarrow \Psi$, namely, for every object $X = (M, F, \alpha)$ in $\langle Q, \rho\rangle$, the morphism

$$E_X : \Phi(X) = (M, \varphi(F), \alpha \circ \sigma) \longrightarrow \Psi(X) = (M, \psi(F), \alpha \circ \tau)$$

in $\langle Q', \rho'\rangle$ given by the identity id_M. Let us check that $\mathrm{id}_M : \Phi(X) \longrightarrow \Psi(X)$ is indeed a morphism in $\langle Q', \rho'\rangle$. We can without loss of generality suppose that Q and Q' are finite quivers. What has to be shown is that the two actions of $\mathrm{End}(\rho')$ on M, one induced by s and the other by t, agree. Indeed, already the two algebra morphisms

$$\sigma, \tau : \mathrm{End}(\rho') \longrightarrow \mathrm{End}(\rho)$$

are the same: given an element $(e_{q'})_{q'\in Q'}$ of $\mathrm{End}(\rho')$ and $q \in Q$, the diagram

$$\begin{tikzcd} \rho(q) \arrow[r, "s_q^{-1}"] \arrow[d, equal] & \rho'(\varphi(q)) \arrow[r, "e_{\varphi(q)}"] \arrow[d, "\eta_q"'] & \rho'(\varphi(q)) \arrow[r, "s_q"] \arrow[d, "\eta_q"'] & \rho(q) \arrow[d, equal] \\ \rho(q) \arrow[r, "t_q^{-1}"] & \rho'(\psi(q)) \arrow[r, "e_{\psi(q)}"] & \rho'(\psi(q)) \arrow[r, "t_q"] & \rho(q) \end{tikzcd}$$

commutes because η_q is not just an arbitrary morphism of modules, but comes from a morphism $\widetilde{\rho}'(\varphi(q)) \longrightarrow \widetilde{\rho}'(\psi(q))$ in $\langle Q', \rho' \rangle$ and hence is $\mathrm{End}(\rho')$-linear.

Theorem 1.10. *Let $\mathbf{A}$ be an abelian, R-linear category, and let $h : \mathbf{A} \longrightarrow \mathbf{Modf}_R$ be a faithful, linear and exact functor. Regard h as a quiver representation. The canonical lift $\widetilde{h} : \mathbf{A} \longrightarrow \langle \mathbf{A}, h\rangle$ is an equivalence of categories.*

References. This was originally shown by Nori in [N]. There are accounts by Bruguières, Levine, and Huber and Müller-Stach ([Bru04, Lev05, HM14]). Ivorra deduces in [Ivo14] the result from a more general construction. $\square$

- 1.11. Theorem 1.10 has the following immediate consequence: Suppose we are given a quiver representation $\rho : Q \longrightarrow \mathbf{Modf}_R$, a faithful and exact functor $h : \mathbf{A} \longrightarrow \mathbf{Modf}_R$ as in 1.10, and a lift $\psi : Q \longrightarrow \mathbf{A}$ of ρ, so that the diagram of solid arrows

$$
\begin{tikzcd}
 & \mathbf{A} \arrow[ddr, "h"] & \\
 & \langle Q,\rho\rangle \arrow[u, dotted] \arrow[dr, "f"] & \\
Q \arrow[uur, "\psi"] \arrow[ur, "\widetilde{\rho}"'] \arrow[rr, "\rho"'] & & \mathbf{Modf}_R
\end{tikzcd}
\qquad
\begin{array}{rl}
f = & \text{forget} \\
\widetilde{\rho} = & \text{canonical lift}
\end{array}
$$

commutes up to a natural isomorphism. We can then regard ψ as a morphism of quiver representations. By naturality of the linear hull construction it gives a functor $\langle Q, \rho\rangle \longrightarrow \langle \mathbf{A}, h\rangle$, or, in view of theorem 1.10, a functor $\langle Q, \rho\rangle \longrightarrow \mathbf{A}$ which renders the whole diagram commutative up to natural isomorphisms. Another consequence of Theorem 1.10, which is in fact a key element in its proof, is the following statement:

Corollary 1.12. *Let $\rho : Q \longrightarrow \mathbf{Modf}_R$ be a quiver representation. Every object of its linear hull $\langle Q, \rho\rangle$ is isomorphic to a subquotient of a sum of objects of the form $\widetilde{\rho}(q)$ for objects q of Q.*

Proof. Apply 1.11 to the full subcategory $\mathbf{A}$ of $\langle Q, \rho\rangle$ consisting of those objects which are subquotients of a sums of objects of the form $\widetilde{\rho}(q)$. $\square$

- 1.13. One might be tempted to replace the category $\mathbf{Modf}_R$ in Definition 1.6 by an arbitrary abelian monoidal closed category. However, this will not result in a useful definition, since Theorem 1.10 and the universal property described in 1.11 do not hold in this generality. The point is the following: Let $\rho : Q \longrightarrow \mathbf{Modf}_R$ be a quiver representation, and regard the forgetful functor $f : \langle Q, \rho\rangle \longrightarrow \mathbf{Modf}_R$ as a quiver representation. The the key turn in the proof of 1.10 is to show that the canonical lift $\widetilde{f}$ of f, and the functor P induced by $\widetilde{\rho} : Q \longrightarrow \langle Q, \rho\rangle$ viewed as a morphism of quiver representations

$$\widetilde{f},\, P \;:\; \langle Q, \rho\rangle \Longrightarrow \langle \langle Q, \rho\rangle \,, f\rangle$$

are isomorphic functors. This relies on the fact that the neutral object for the tensor product in $\mathbf{Modf}_R$ is a projective generator, which is particular to categories of modules. In the case where we replace $\mathbf{Modf}_R$ by a tannakian category, a correct abelian hull which satisfies Ivorra's universal property (it is the initial object in a certain strict 2-category, see [Ivo14], Definition 2.2) is given by the equaliser category of $\widetilde{f}$ and P.

Lemma 1.14. *Let $\psi : (Q \xrightarrow{\rho} \mathbf{Modf}_R) \longrightarrow (Q' \xrightarrow{\rho'} \mathbf{Modf}_R)$ be a morphism of quiver representations. The induced functor $\Psi : \langle Q, \rho\rangle \longrightarrow \langle Q', \rho'\rangle$ is an equivalence of categories if and only if there exists a quiver representation $\lambda : Q' \longrightarrow \langle Q, \rho\rangle$ such that the following diagram commutes up to natural isomorphisms.*

$$\begin{tikzcd} Q \arrow[r, "\widetilde{\rho}"] \arrow[d, "\psi"'] & \langle Q, \rho\rangle \arrow[d, "\Psi"] \\ Q' \arrow[r, "\widetilde{\rho}'"'] \arrow[ur, "\lambda"] & \langle Q', \rho'\rangle \end{tikzcd} \tag{4}$$

Proof. If Ψ is an equivalence of categories, then there exists a functor $\Phi : \langle Q', \rho'\rangle \longrightarrow \langle Q, \rho\rangle$ and isomorphisms $\Phi \circ \Psi \cong \mathrm{id}$ and $\Psi \circ \Phi \cong \mathrm{id}$. A possible choice for λ is then $\lambda := \Phi \circ \widetilde{\rho}'$, indeed, since the outer square in (4) commutes up to an isomorphism, we have isomorphisms

$$\Psi \circ \lambda \cong \Psi \circ \Phi \circ \widetilde{\rho}' \cong \widetilde{\rho}' \qquad \text{and} \qquad \lambda \circ \psi = \Phi \circ \widetilde{\rho}' \circ \psi \cong \Phi \circ \Psi \circ \widetilde{\rho} \cong \widetilde{\rho}$$

as required. On the other hand, suppose that a representation λ as in the statement of the lemma exists. We extend the diagram (4) to a diagram

$$\begin{tikzcd} Q \arrow[r, "\widetilde{\rho}"] \arrow[d, "\psi"'] & \langle Q, \rho\rangle \arrow[d, "\Psi"] \\ Q' \arrow[r, "\widetilde{\rho}'"] \arrow[ur, "\lambda"] \arrow[d, "\lambda"'] & \langle Q', \rho'\rangle \arrow[d, "\Lambda"] \\ \langle Q, \rho\rangle \arrow[r, "P"'] \arrow[ur, "\Psi"] & \langle \langle Q, \rho\rangle, f\rangle \end{tikzcd} \tag{5}$$

with arrows as follows: Let $f : \langle Q, \rho\rangle \longrightarrow \mathbf{Modf}_R$ and $f' : \langle Q', \rho'\rangle \longrightarrow \mathbf{Modf}_R$ be the forgetful functors. We have an isomorphism $f' \circ \Psi \cong f$, hence an isomorphism $\rho' = f' \circ \widetilde{\rho}' \cong f' \circ \Psi \circ \lambda \cong f \circ \lambda$, and can thus view λ as a morphism of quiver representations from ρ' to f. The arrow Λ is the corresponding functor. The functor P is the canonical lift of $\widetilde{\rho}$ regarded as a morphism of quiver representations from ρ to f; by Theorem 1.10 it is an equivalence of categories. Let ι be a quasi-inverse to P. I claim that the functor $\Phi := \iota \circ \Lambda$ is a quasi-inverse to Ψ.
To get an isomorphism $\Phi \circ \Psi \cong \mathrm{id}$ it suffices to get an isomorphism $\Lambda \circ \Psi \cong P$. Let us apply 1.9 to the representations

$$\begin{tikzcd} Q \arrow[r, shift left, "\lambda\circ\psi"] \arrow[r, shift right, "\widetilde{\rho}"'] \arrow[dr, "\rho"'] & \langle Q, \rho\rangle \arrow[d, "f"] \\ & \mathbf{Modf}_R \end{tikzcd} \qquad\qquad \begin{array}{l} f \circ \lambda \circ \psi \cong f \circ \widetilde{\rho} = \rho \\[1em] f \circ \widetilde{\rho} = \rho \end{array}$$

where we use an isomorphism $\lambda \circ \psi \cong \widetilde{\rho}$ which makes (4) commute. This isomorphism induces an isomorphism $\eta : \widetilde{f} \circ \lambda \circ \psi \cong \widetilde{f} \circ \widetilde{\rho}$ which makes the diagrams corresponding to (3) commute, hence we obtain an isomorphism of functors $\Lambda \circ \Psi \cong P$. It remains to construct an isomorphism $\Psi \circ \Phi \cong \mathrm{id}$. This is done by replacing the diagram (4) in the statement of the Lemma with the bottom half of (5), and the same application of 1.9. $\square$

- 1.15 (Caveat). In the situation of Lemma 1.14, it will not do to just produce a representation λ as in diagram (4) and natural isomorphisms of R-modules $\rho(q) \cong f(\lambda(q))$ in order to show that Ψ is an equivalence. Such a λ will produce *some* functor $\Phi : \langle Q', \rho' \rangle \longrightarrow \langle Q, \rho \rangle$ which, in general, is not a quasi-inverse to Ψ. Each time we apply 1.14, the hard part is not to define λ, but to check commutativity of the diagram. The point seems to have been overlooked at several places. Consider for example a homomorphism of finite groups $G' \longrightarrow G$, the quivers Q and Q' of finite G-sets, respectively G'-sets, and the quiver representations ρ and ρ' which associate with a set X the free R-module generated by X. The linear hulls identify with the categories of group representations in finitely presented R-modules, and the restriction functor $Q \longrightarrow Q'$ is a morphism of quiver representations which induces the restriction functor between representation categories. For any G'-set X' write $\lambda(X)$ for the constant G-representation on the free R-module generated by the set X. We obtain a diagram

$$\begin{tikzcd} G\,\mathbf{Set} \arrow[r, "\text{free}"] \arrow[d, "\psi=\text{res}"'] & \mathbf{Rep}_R(G) \arrow[d, "\Psi=\text{res}"] \\ G'\,\mathbf{Set} \arrow[r, "\text{free}"'] \arrow[ur, "\lambda"] & \mathbf{Rep}_R(G') \end{tikzcd}$$

which does in not commute except in trivial cases, but commutes after forgetting the group actions. The functor Ψ is not an equivalence, trivial cases excepted, and the functor induced by λ sends a G'-representation V to the constant G-representation with underlying module V.

1.3. Products of quiver representations.

Definition 1.16. Let $\rho : Q \longrightarrow \mathbf{Modf}_R$ and $\rho' : Q' \longrightarrow \mathbf{Modf}_R$ be quiver representations. We denote by

$$\rho \boxtimes \rho' : Q \boxtimes Q' \longrightarrow \mathbf{Modf}_R$$

the following quiver representation. Objects of the quiver $Q \boxtimes Q'$ are pairs (q, q') consisting of an object q of Q and an object q' of Q', and morphisms are either of the form $(\mathrm{id}_q, f') : (q, q') \longrightarrow (p, p')$ for some morphism $f' : q' \longrightarrow p'$ in Q', or of the form $(f, \mathrm{id}_{q'}) : (q, q') \longrightarrow (p, q')$ for some morphism $f : p \longrightarrow q$ in Q. The representation $\rho \boxtimes \rho'$ is defined by

$$(\rho \boxtimes \rho')(q, q') = \rho(q) \otimes_R \rho'(q')$$

on objects, and by $(\rho \boxtimes \rho')(\mathrm{id}_q, f') = \mathrm{id}_{\rho(q)} \otimes \rho(f')$ and $(\rho \boxtimes \rho')(f, \mathrm{id}_{q'}) = \rho(f) \otimes \mathrm{id}_{\rho'(q')}$ on morphisms.

- 1.17. Our next proposition relates the linear hull of a quiver representation $\rho \boxtimes \rho'$ with the tensor product of the linear hulls of ρ and ρ'. A tensor product $\mathbf{A} \otimes_R \mathbf{B}$ of abelian R-linear categories $\mathbf{A}$ and $\mathbf{B}$, as introduced in [Del90], is an R-linear category characterised up to equivalence by a universal property. It does not exist in general as is shown in [Lop13], however, it exists and has good properties as soon as one works in an appropriate enriched setting, as is shown in [Gre10]. We only need to know the following fact: If $\mathbf{A}$ is the category of continuous A-modules and $\mathbf{B}$ the category continuous of B-modules in $\mathbf{Modf}_R$ for some R-proalgebras A and B, then $\mathbf{A} \otimes_R \mathbf{B}$ exists and is given by the category of continuous $A \otimes_R B$-modules. This follows from §5.1 and Proposition 5.3 of [Del90].

Proposition 1.18. *There is a canonical faithful and exact functor*

$$\langle Q \boxtimes Q', \rho \boxtimes \rho' \rangle \longrightarrow \langle Q, \rho \rangle \otimes_R \langle Q', \rho' \rangle$$

which commutes with the forgetful functors to $\mathbf{Modf}_R$*, and is natural in* ρ *and* ρ' *for morphisms of quiver representations. This functor is an equivalence of categories if* R *is hereditary*[2] *and* $\rho(q)$ *and* $\rho'(q')$ *are projective* R*-modules for all* $q \in Q$ *and* $q' \in Q'$*.*

Proof. It suffices to construct an R-algebra homomorphism $\mathrm{End}(\rho) \otimes_R \mathrm{End}(\rho') \longrightarrow \mathrm{End}(\rho \boxtimes \rho')$ in the case where Q and Q' are finite quivers. Set

$$V := \bigoplus_{q \in Q} \rho(q)$$

and write $X \subseteq \mathrm{End}_R(V)$ for the finite set of compositions of the form $V \xrightarrow{\mathrm{pr}} \rho(p) \xrightarrow{\rho(f)} \rho(q) \xrightarrow{\subseteq} V$ for some morphism f in Q, and write $E_X := \mathrm{End}(\rho) \subseteq \mathrm{End}(V)$ for the commutator of X. Define $E_{X'} \subseteq \mathrm{End}(V')$ and $E_{X \boxtimes X'} \subseteq \mathrm{End}(V \otimes V')$ similarly. The canonical, natural morphism of R-algebras

$$\alpha : E_X \otimes_R E_{X'} \longrightarrow E_{X \boxtimes X'} \qquad\qquad \alpha(f \otimes f')(v \otimes v') = f(v) \otimes f'(v')$$

yields the functor we sought. It remains to show that this morphism is an isomorphism if R is hereditary and if V and V' are projective. If V is projective then so is $\mathrm{End}(V)$, and there is an exact sequence

$$0 \longrightarrow E_X \longrightarrow \mathrm{End}(V) \xrightarrow{\delta} \prod_{x \in X} \mathrm{End}(V)$$

where δ is given by $\delta(f)_x = x \circ f - f \circ x$. Under the hypothesis on R the image of δ is projective, and hence E_X is a direct factor of $\mathrm{End}(V)$. The same observation applies to $E_{X'}$ and $E_{X \boxtimes X'}$, and it follows that all morphisms in the diagram

$$\begin{array}{ccc} E_X \otimes_R E_{X'} & \xrightarrow{\alpha} & E_{X \boxtimes X'} \\ \downarrow & & \downarrow \subseteq \\ \mathrm{End}(V) \otimes_R \mathrm{End}(V') & \xrightarrow{\beta} & \mathrm{End}(V \otimes V') \end{array} \qquad \beta(f \otimes f')(v \otimes v') = f(v) \otimes f'(v') \tag{6}$$

are injective, and the bottom horizontal map β is an isomorphism. It remains to show that the top horizontal map is surjective. Let $f \in \mathrm{End}(V \otimes V')$ be an endomorphism that commutes with $X \boxtimes X'$. We write f as $f = \beta(f_1 \otimes f_1' + \cdots + f_n \otimes f_n')$ with $f_i \in \mathrm{End}(V)$ and linearly independent $f_i' \in \mathrm{End}(V')$. For all $x \in X$ we have $(x \otimes 1) \circ f = f \circ (x \otimes 1)$, that is,

$$\sum_{i=1}^{n} (x \circ f_i - f_i \circ x) \otimes f_i' = 0$$

and hence $f_i \in E_X$. In other words, f comes via β from an element of $E_X \otimes \mathrm{End}(V')$, and symetrically, f comes from an element of $\mathrm{End}(V) \otimes E_{X'}$. Finally, again since E_X and $E_{X'}$ are direct factors of $\mathrm{End}(V)$ and $\mathrm{End}(V')$, we have

$$\big(E_X \otimes \mathrm{End}(V')\big) \cap \big(\mathrm{End}(V) \otimes E_{X'}\big) = E_X \otimes_R E_{X'}$$

so $\beta^{-1}(f)$ is indeed an element of $E_X \otimes_R E_{X'}$ as we wanted to show. □

[2] *hereditary* means: Every ideal of R is projective, or equivalently, every submodule of a projective module is projective. Fields, finite products of fields, Dedekind rings and finite rings are examples. A commutative, coherent and hereditary ring which has no zero divisors is either a field or a Dedekind ring.

1.4. **Change of coefficients.**

- 1.19. At last, we have to discuss what happens with linear hulls of quiver representations when we change the coefficient ring. Given a quiver representation $\rho : Q \longrightarrow \mathbf{Modf}_R$ and a commutative R-algebra S, we define a representation $\rho_S : Q \longrightarrow \mathbf{Modf}_S$ by

$$\rho_S(q) = S \otimes_R \rho(q) \qquad \text{and} \qquad \rho(f) = \mathrm{id}_S \otimes f$$

on objects q and morphisms f of Q. There is a canonical S-linear morphism of proalgebras

$$\mathrm{End}_R(\rho) \otimes_R S \longrightarrow \mathrm{End}(\rho_S) \tag{7}$$

inducing an S-linear faithful functor

$$\langle Q, \rho_S\rangle \longrightarrow \langle Q, \rho\rangle \otimes \mathbf{Modf}_S \tag{8}$$

The morphism (7) is in general not an isomorphism, and hence (8) is generally not an equivalence of categories.

Proposition 1.20. *Under each one of the following conditions, the change of coefficients functor (8) is an equivalence of categories.*

(1) *As an R-module, S is projective.*
(2) *The ring R is a Dedekind domain, and S is a localisation or a completion of R.*

Proof. It suffices to show that the algebra morphism $\mathrm{End}_R(\rho)\otimes_R S \longrightarrow \mathrm{End}(\rho_S)$ is an isomorphism for every finite quiver Q and representation $\rho : Q \longrightarrow \mathbf{Modf}_R$. Under each of the given conditions, the natural morphism

$$\mathrm{Hom}_R(M, N) \otimes S \cong \mathrm{Hom}_S(M \otimes S, N \otimes S)$$

sending $f \otimes s$ to the map $m \otimes t \longmapsto f(m) \otimes st$ is an isomorphism. Because S is flat, the upper horizontal row in

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{End}(\rho)\otimes S \arrow[r] \arrow[d] & \prod_{q\in Q} \mathrm{End}(\rho(q)) \otimes S \arrow[r] \arrow[d, "\cong"] & \prod_{p\to q} \mathrm{Hom}(\rho(p), \rho(q)) \otimes S \arrow[d, "\cong"] \\
0 \arrow[r] & \mathrm{End}(\rho_S) \arrow[r] & \prod_{q\in Q} \mathrm{End}(\rho(q) \otimes S) \arrow[r] & \prod_{p\to q} \mathrm{Hom}(\rho(p) \otimes S, \rho(q) \otimes S)
\end{tikzcd}$$

is exact, hence the claim. $\square$

Proposition 1.21. *Suppose that R is hereditary, and that $\rho : Q \longrightarrow \mathbf{Modf}_R$ is a quiver representation such that $\rho(q)$ is projective for every $q \in Q$. Then, the change of coefficients morphism (7) is injective for every R-algebra S.*

Proof. We may suppose that that Q is finite. Under the hypotheses on R and ρ, the image of δ in the sequence

$$0 \longrightarrow \mathrm{End}(\rho) \longrightarrow \prod_{q\in Q} \mathrm{End}(\rho(q)) \xrightarrow{\ \delta\ } \prod_{p\to q} \mathrm{Hom}(\rho(p), \rho(q))$$

is projective, hence the inclusion of $\mathrm{End}(\rho)$ has a retraction. It follows that the induced morphism obtained by tensoring with S is still injective. The claim follows by considering the same diagram as in the proof of Proposition 1.20. This time, the two right hand vertical maps are isomorphisms because the $\rho(q)$ are projective. $\square$

- 1.22. In terms of the functor (8), the conclusion of Proposition 1.21 is as follows: *Every object of $\langle Q, \rho\rangle \otimes \mathbf{Modf}_S$ is isomorphic to a subquotient of an object from $\langle Q, \rho_S\rangle$.*

1.5. **Examples.**

Example 1.23. Let G be a finite group, and let Q be the category of finite G-sets, viewed as a quiver. Let $\rho : Q \longrightarrow \mathbf{Modf}_R$ be the quiver representation sending a finite G-set X to the free R-module $\rho(X) = R^X$ generated by X, and morphisms of G-sets $X \longrightarrow Y$ to the induced R-linear map $R^X \longrightarrow R^Y$. We can consider G as a G-set, hence obtain a morphism $\mathrm{End}(\rho) \longrightarrow \mathrm{End}_R(R^G)$ whose image must commute with all maps $R^G \longrightarrow R^G$ induced by right multiplication with an element of G. This morphism is thus a morphism

$$\mathrm{End}(\rho) \longrightarrow R[G]$$

where $R[G]$ is the group algebra of G. It is not hard to check that this morphism is indeed an isomorphism, hence the linear hull $\langle Q, \rho\rangle$ is the category of finitely presented R-modules with a linear G-action.

Example 1.24. Let Q be the quiver whose objects are the integers $Q = \{\ldots, -2, -1, 0, 1, \ldots\}$ and whose morphisms are just the identities. Let $\rho : Q \longrightarrow \mathbf{Modf}_R$ be the quiver representation sending each object $n \in Q$ to the R-module R. Endomorphisms of ρ are sequences $(r_n)_{n\in\mathbb{Z}}$ of elements of R, and form a ring by component wise multiplication. Hence, the linear hull $\langle Q, \rho\rangle$ is canonically equivalent to the category of finitely presented, $\mathbb{Z}$-graded R-modules. If R is a field, then this category is also the category of finite dimensional representations of the multiplicative group $\mathbb{G}_m$, or equivalently, the category of finite dimensional comodules of the coalgebra $R[t, t^{-1}]$. This coalgebra is indeed the continuous R-linear dual of the proalgebra $\mathrm{End}(\rho)$.

Example 1.25. Here is a slightly pathological but not irrelevant example. Let Q be the quiver whose objects are $\{a, b, q_1, q_2, q_3, \ldots\}$ and whose morphisms are besides the identities one morphisms $a \longrightarrow q_n$ for each n, one morphism $b \longrightarrow q_n$ for each n, and one morphism $q_m \longrightarrow q_n$ if m is a multiple of n. Consider the quiver representation $\rho : Q \longrightarrow \mathbf{Modf}_{\mathbb{Z}}$ given by

$$\rho(a) = \mathbb{Z} \qquad \text{and} \qquad \rho(b) = \mathbb{Z} \qquad \text{and} \qquad \rho(q_n) = \mathbb{Z}/n\mathbb{Z}$$

on objects, and by reduction modulo n on morphisms with target q_n. For $N \geq 1$, let $Q_N \subseteq Q$ be the full subquiver of Q containing a, b and $q_1 \ldots, q_N$. It is a finite subquiver, and to give an endomorphism of $\rho|_{Q_N}$ is to give a pair of integers (e_a, e_b) satisfying $e_a \equiv e_b \bmod N$. By definition, the linear hull $\langle Q, \rho\rangle$ is the category of continuous modules of the proalgebra

$$\mathrm{End}(\rho) := \lim_N \mathrm{End}(\rho|_{Q_N})$$

with the hereby given pro-structure. It is not hard to show that the category $\langle Q,\rho\rangle$ is indeed canonically equivalent to the category whose objects are finitely generated $\mathbb{Z}$-modules M together with a decomposition $M\otimes\mathbb{Q}=V_a\oplus V_b$ of rational vector spaces. It is somehow tempting to "compute" $\mathrm{End}(\rho)$ - an element of $\mathrm{End}(\rho)$ would be a sequence of pairs of integers which is compatible for transition maps, thus in fact a constant sequence given by a pair (e_a,e_b) satisfying $e_a\equiv e_b \bmod N$ for *all* integers N, so just a single integer. Indeed, the ring of endomorphisms of ρ is $\mathbb{Z}$, but this is not what we mean by writing $\mathrm{End}(\rho)$, and gives the wrong category of modules. One notices that a nonzero group homomorphism $\rho(a)=\mathbb{Z}\longrightarrow\mathbb{Z}=\rho(b)$ is linear for the "wrong" ring of endomorphisms of ρ, but not $\mathrm{End}(\rho|_{Q_N})$-linear for any $N\geq 1$, so not $\mathrm{End}(\rho)$-linear, hence $\widetilde{\rho}(a)$ and $\widetilde{\rho}(b)$ are nonisomorphic objects in $\langle Q,\rho\rangle$.

2. Nori's category of mixed motives

In this section, we define the category of mixed Nori motives over a subfield k of $\mathbb{C}$, and introduce realisation functors. Nori's original definition of his category of mixed cohomological motives is slightly different from ours. The most important difference is that Nori first introduces a category of *effective* motives and then inverts the motive $H^1(\mathbb{G}_m)$ with respect to a monoidal structure, which we will introduce much later. The idea to build in Tate twists from the start I have taken from the introduction to [Ayo14a]. The advantage of Ayoub's modification is that we get immediately the right category and don't have to introduce an effective variant and tensor products to introduce it.

- 2.1. We fix for this section a subfield k of the field of complex numbers $\mathbb{C}$. All varieties are understood to be quasiprojective varieties (separated schemes of finite type) over k. We keep the assumptions of 1.1, in particular the coefficient ring R is a commutative coherent ring.

Definition 2.2. We call *quiver of relative varieties over* k and denote by $\mathrm{Q}(k)$ the quiver whose objects and morphisms are as follows:

(1) Objects are quadruples $[X,Y,n,i]$, where Y is a closed subvariety of a quasiprojective variety X over k, and where n and i are integers.
(2) Morphisms with target $[X,Y,n,i]$ are given by either (a), (b) or (c) as follows:
   (a) a morphism $[X',Y',n,i]\longrightarrow[X,Y,n,i]$ for every morphism of varieties $f:X\longrightarrow X'$ satisfying $f(Y)\subseteq Y'$.
   (b) a morphism $[Y,Z,n-1,i]\longrightarrow[X,Y,n,i]$ for every pair of closed immersions $X\supseteq Y\supseteq Z$
   (c) a morphism $[X\times\mathbb{G}_m,(Y\times\mathbb{G}_m)\cup(X\times\{1\}),n+1,i+1]\longrightarrow[X,Y,n,i]$.

- 2.3. We denote by $\rho:\mathrm{Q}(k)\longrightarrow\mathbf{Modf}_R$ the quiver representation that sends an object $[X,Y,n,i]$ to the R-module

$$\rho([X,Y,n,i])=H^n([X,Y],R)(i)$$

where $H^n([X,Y],R)$ is the relative cohomology of the pair of topological spaces $Y(\mathbb{C}) \subseteq X(\mathbb{C})$, and the twist (i) means tensoring $|i|$ times with the R-module $H^1(\mathbb{G}_{m,k}(\mathbb{C}),R)$ for negative i, or its R-linear dual if i is positive[3]. Using the notations of Definition 2.2, a morphism $[X',Y',n,i] \longrightarrow [X,Y,n,i]$ of type (a) is sent to the morphism of R-modules

$$H^n([X',Y'],R)(i) \longrightarrow H^n([X,Y],R)(i)$$

obtained from f by functoriality of cohomology. A morphism of type (b) is sent to the connecting morphism

$$H^{n-1}([Y,Z],R)(i) \longrightarrow H^n([X,Y],R)(i)$$

in the long exact sequence of modules obtained from the triple $Z(\mathbb{C}) \subseteq Y(\mathbb{C}) \subseteq X(\mathbb{C})$. Finally, a morphism of type (c) is sent to the isomorphism

$$H^{n+1}([X\times\mathbb{G}_m,(Y\times\mathbb{G}_m)\cup(X\times\{1\})],R) \longrightarrow H^n([X,Y],R)\otimes H^1([\mathbb{G}_m,\{1\}],R)$$

given by the Künneth formula, twisted by $H^1(\mathbb{G}_m,R)^{\otimes(-i-1)}$.

Definition 2.4. We call *category of mixed motives over k with coefficients in R* the category

$$\mathbf{M}(k)_R := \langle \mathrm{Q}(k),\rho\rangle$$

and denote by $H^n([X,Y])(i)$ the image of $[X,Y,n,i]$ in $\mathbf{M}(k)$ under the canonical lift $\widetilde{\rho}$. If the coefficients R or the ground field k are clear from the context, we drop the corresponding symbols from the notation $\mathbf{M}(k)_R$.

Proposition 2.5. (1) *The category $\mathbf{M}(k)_R$ is abelian and R-linear.*
(2) *Homomorphism sets in $\mathbf{M}(k)_R$ are finitely presented R-modules.*
(3) *The forgetful functor $\mathbf{M}(k)_R \longrightarrow \mathbf{Modf}_R$ is exact and faithful, and in particular reflects isomorphisms.*
(4) *If a morphism of relative varieties $[X',Y'] \longrightarrow [X,Y]$ induces a homotopy equivalence between relative topological spaces $[X'(\mathbb{C}),Y'(\mathbb{C})] \longrightarrow [X(\mathbb{C}),Y(\mathbb{C})]$, then it induces isomorphisms of motives $H^n([X,Y])(i) \longrightarrow H^n([X',Y'])(i)$.*

Proof. The first three statements are true for the R-linear hull of any quiver representation. Hence the last statement: such a morphism of relative varieties induces an isomorphism of cohomology groups $H^n(X(\mathbb{C}),Y(\mathbb{C}))(i) \longrightarrow H^n(X'(\mathbb{C}),Y'(\mathbb{C}))(i)$. □

- 2.6. We denote by $\mathrm{R_B} : \mathbf{M}(k)_R \longrightarrow \mathbf{Modf}_R$ the forgetful functor and call it *Betti realisation*. We will now define other realisation functors: the *Hodge realisation* $\mathrm{R_{Hdg}}$ and the ℓ-adic realisations R_ℓ. The Hodge realisation is a functor from $\mathbf{M}(k)_\mathbb{Z}$ to the category $\mathbf{MHS}_\mathbb{Z}$ of integral mixed Hodge structures. By an integral Hodge structure we understand a finitely generated and not necessarily torsion free $\mathbb{Z}$-module L together with a rational mixed Hodge structure on $L\otimes\mathbb{Q}$. For every prime number ℓ, the ℓ-adic realisation functor R_ℓ is a functor from $\mathbf{M}(k)_{\mathbb{Z}_\ell}$ to the category of continuous

[3]The weight convention this will lead to is the one in [And04]: Hodge structures in the cohomology of varieties have positive weights, so $H^2(\mathbb{P}^1(\mathbb{C}),\mathbb{Q}) = \mathbb{Q}(-1) = (2\pi i)^{-1}\mathbb{Q} \subseteq \mathbb{C}$ is a pure Hodge structure of type $(1,1)$ and weight 2, and $H^2_{\text{ét}}(\mathbb{P}^1,\mathbb{Q}_\ell) = \mathbb{Q}_\ell(-1)$ is dual to $\mathbb{Q}_\ell(1) = \lim \mu_{\ell^n}$.

$\mathbb{Z}_\ell$-linear representations of the absolute Galois group $\mathrm{Gal}(\overline{k}|k)$, where $\overline{k}$ is the algebraic closure of k in $\mathbb{C}$.

Corollary 2.7 (to Theorem 1.10). *There exists a unique faithful, exact and $\mathbb{Z}$-linear functor $\mathrm{R}_{\mathrm{Hdg}}$ such that the diagram*

$$\begin{tikzcd} & \mathrm{Q}(k) \arrow[dl, "\widetilde{\rho}"'] \arrow[dr, "\lambda"] & \\ \mathbf{M}(k)_{\mathbb{Z}} \arrow[rr, "\mathrm{R}_{\mathrm{Hdg}}"] \arrow[dr, "\mathrm{R}_{\mathrm{B}}"'] & & \mathbf{MHS}_{\mathbb{Z}} \arrow[dl, "\mathrm{forget}"] \\ & \mathbf{Modf}_{\mathbb{Z}} & \end{tikzcd}$$

commutes. In this diagram, λ is the quiver representation which sends objects $[X, Y, n, i]$ to the mixed Hodge structure which enriches the cohomology group $H^n([X, Y], \mathbb{Z})(i)$, and morphisms to the corresponding morphisms of mixed Hodge structures.

Proof. The statement follows from Theorem 1.10 and its consequences stated in 1.11, applied to the lift λ of ρ which is given in the statement of the Corollary. The construction of the mixed Hodge structure enriching $H^n([X, Y], \mathbb{Z})(i)$ is due to Deligne. For every morphism f of $Q(k)$, the morphism $\mathbb{Z}$-modules $\rho(f)$ is compatible with these Hodge structures. □

Corollary 2.8 (to Theorem 1.10). *Let $\overline{k}$ be the algebraic closure of k in $\mathbb{C}$, and let $\mathbf{Rep}(\mathrm{Gal}_k)_{\mathbb{Z}_\ell}$ be the category of continuous representations of $\mathrm{Gal}_k := \mathrm{Gal}(\overline{k}|k)$ in finitely generated $\mathbb{Z}_\ell$-modules with their ℓ-adic topology. There exists a unique faithful, exact and $\mathbb{Z}_\ell$-linear functor R_ℓ such that the diagram*

$$\begin{tikzcd} & \mathrm{Q}(k) \arrow[dl, "\widetilde{\rho}"'] \arrow[dr, "\lambda"] & \\ \mathbf{M}(k)_{\mathbb{Z}_\ell} \arrow[rr, "\mathrm{R}_\ell"] \arrow[dr, "\mathrm{R}_{\mathrm{B}}"'] & & \mathbf{Rep}(\mathrm{Gal}_k)_{\mathbb{Z}_\ell} \arrow[dl, "\mathrm{forget}"] \\ & \mathbf{Modf}_{\mathbb{Z}_\ell} & \end{tikzcd}$$

commutes. In this diagram, λ sends the objects $[X, Y, n, i]$ to the ℓ-adic étale cohomology groups $H^n([X_{\overline{k}}, Y_{\overline{k}}], \mathbb{Z}_\ell)(i)$ equipped with their Galois action, and morphisms to the morphisms given by the various functorialities of ℓ-adic cohomology. The same statement holds for coefficient rings $\mathbb{Z}/\ell^m\mathbb{Z}$ in place of $\mathbb{Z}_\ell$.

Proof. Let me recall how the relative étale cohomology groups are defined. Let $\beta : U \longrightarrow X$ be the inclusion of the complement of Y, and denote by $\mathbb{Z}/\ell^m$ the constant sheaf with value $\mathbb{Z}/\ell^m$ on $(X_{\overline{k}})_{\text{ét}}$, the small étale site on $X_{\overline{k}}$. We set

$$H^n([X_{\overline{k}}, Y_{\overline{k}}], \mathbb{Z}/\ell^m) := \lim_m H^n((X_{\overline{k}})_{\text{ét}}, \beta_!\beta^*\mathbb{Z}/\ell^m)$$

where , and $\beta_!$ and β^* are sheaf opreations for sheaves on étale sites. These are finite $\mathbb{Z}/\ell^m$-modules, which depend functorially on $[X, Y]$, hence the Galois group Gal_k acts continuously on them. The

ℓ-adic étale cohomology groups $H^n([X_{\overline{k}}, Y_{\overline{k}}], \mathbb{Z}_\ell)(i)$ are obtained by taking limits over $m \geq 0$. They are finitely generated $\mathbb{Z}_\ell$-modules equipped with a continuous Galois action.

So far, we have explained the quiver representation λ on objects. The definition of λ on morphisms of type (a) and (c) (Definition 2.2) is clear. For morphisms of type (b), notice that a triple $Z \subseteq Y \subseteq X$, we obtain a short exact sequence of sheaves on $(X_{\overline{k}})_{\text{ét}}$ whose associated long exact sequence reads

$$\cdots \longrightarrow H^{n-1}([X_{\overline{k}}, Z_{\overline{k}}], \mathbb{Z}/\ell^m)(i) \longrightarrow H^{n-1}(Y_{\overline{k}}, Z_{\overline{k}}], \mathbb{Z}/\ell^m)(i) \xrightarrow{\partial} H^n([X_{\overline{k}}, Y_{\overline{k}}], \mathbb{Z}/\ell^m)(i) \longrightarrow \cdots$$

and λ sends the morphism of type (b) with target $[X, Y, n, i]$ to the connecting morphism ∂.

In order to deduce the corollary at hand from Theorem 1.10, we need to produce natural isomorphisms of $\mathbb{Z}_\ell$-modules

$$\underset{\text{étale cohomology}}{H^n([X_{\overline{k}}, Y_{\overline{k}}], \mathbb{Z}_\ell)(i)} \cong \underset{\text{singular cohomology}}{H^n([X(\mathbb{C}), Y(\mathbb{C})], \mathbb{Z}_\ell)(i)} = \rho([X, Y, n, i])$$

where naturality refers to morphisms in $Q(k)$. Every constructible torsion sheaf $\mathcal{F}$ on X can be viewed as a sheaf for the étale topology. Artin's comparison theorem (SGA 4, exposé XVI, Théorème 4.1) states that there is a canonical and natural isomorphism of finite groups

$$\underset{\text{singular cohomology}}{H^n(X(\mathbb{C}), \mathcal{F})} \quad \cong \quad \underset{\text{étale cohomology}}{H^n(X_{\mathbb{C}}, \mathcal{F})}$$

and the right hand group is isomorphic to $H^n(X_{\overline{k}}, \mathcal{F})$ via base change under the inclusion $\overline{k} \longrightarrow \mathbb{C}$ (SGA 4, exposé XVI, Corollaire 1.6). This settles the corollary in the case of finite coefficients. For $\mathbb{Z}_\ell$-coefficients, it remains to observe that the canonical maps

$$H^n([X, Y], \mathbb{Z}_\ell) \longleftarrow H^n([X, Y], \mathbb{Z}) \otimes \mathbb{Z}_\ell \longrightarrow \lim_m H^n([X, Y], \mathbb{Z})/\ell^m \longrightarrow \lim_m H^n([X, Y], \mathbb{Z}/\ell^m)$$

are all isomorphisms, because all cohomology groups $H^n([X, Y], \mathbb{Z})$ are finitely generated $\mathbb{Z}$-modules, and $\mathbb{Z}_\ell$ is a flat $\mathbb{Z}$-algebra. $\square$

3. Cell decomposition

In a wide range of contexts, spectral sequences associated with simplicial or filtered spaces are a powerful tool when it comes to computing cohomology. Naturally, we would like to use these techniques to compute motives, after all, Nori's motives are but a universal cohomology theory. The problem we face is that the motive $H^n([X, Y], R)(i)$ is not defined as the homology in degree n of a complex as it is the case, for example, with singular cohomology or Galois cohomology. Our first goal in this section is to fabricate adequately functorial complexes which compute motives, as it is done in [N].

We start by recalling a theorem from [N], which provides a sort of cell decomposition for relative algebraic varieties, with properties analogous to a cell decomposition of a CW complex.

- 3.1. As in the previous sections, R is a commutative coherent ring, and k is a subfield of the field of complex numbers $\mathbb{C}$. All R-modules are finitely presented, and all sheaves are sheaves of R-modules. By a variety of dimension $\leq d$ over k we understand a quasiprojective variety over k such that each of its irreducible components is of dimension at most d over k.

Definition 3.2. Let X be a variety over k, let $Y \subseteq X$ be a closed subvariety and let $d \geq 0$ be an integer. We say that the pair $[X,Y]$ is *cellular in degree d* if $H^n([X,Y],R)$ is a projective R-module for $n = d$, and zero for $n \neq d$.

Theorem 3.3 (Cell decomposition)**.** *Let X be an affine variety of dimension $\leq d$ over k and let $(Y_i \longrightarrow X_i \longrightarrow X)_{i\in I}$ be a finite family of closed immersions. There exists a closed subvariety $Z \subseteq X$ of dimension $\leq d-1$, such that each pair $[X_i, Y_i \cup (X_i \cap Z)]$ is cellular in degree d.*

- 3.4. The Cell decomposition Theorem states that, in a sense, affine algebraic varieties admit many cell decompositions. The main ingredient is the following Lemma:

Lemma (Nori's Basic Lemma)**.** *Let X be an affine variety over k of dimension d, and let F be a constructible sheaf on X whose stalks are projective R-modules. There exists a closed (possibly empty) subvariety Z of X of dimension $\leq d-1$, such that*

$$H^n([X,Z],F) = \begin{cases} 0 & \textit{if } n \neq d \\ \textit{a projective } R\textit{-module} & \textit{if } n = d \end{cases}$$

holds.

This Lemma is independently due to Nori and to Beilinson ([Bei87], Lemma 3.3 and some interpretation work). Proofs can be found in [HM16].

Proof of the cell decomposition theorem. If X is of dimension $< d$, we choose $Z = X$ and are done. Otherwise, if X is of dimension d, let us write F_i for the constructible sheaf on X supported on X_i, which computes the cohomology of the pair $[X_i, Y_i]$. Nori's Basic Lemma applied to the direct sum of the F_i implies that there exists a subvariety Z of X such that

$$H^n([X,Z],F_i) = \begin{cases} 0 & \text{if } n \neq d \\ \text{a projective } R\text{-module} & \text{if } n = d \end{cases}$$

holds. But we have $H^n([X,Z],F_i) = H^n([X_i, Y_i \cup (Z \cap X_i)], R)$ by definition of the cohomology of a pair, hence the closed subvariety Z already has the required properties. $\square$

- 3.5. We write $\mathrm{Q_c}(k)$ for the full subquiver of $\mathrm{Q}(k)$ whose objects are those $[X,Y,n,i]$ where X is affine of dimension $\leq n$ and $[X,Y]$ is cellular in degree n. We equip $Q(k)$ with the representation ρ introduced in 2.3, and restrict this representation to $\mathrm{Q_c}(k)$, so that the inclusion $\mathrm{Q_c}(k) \subseteq \mathrm{Q}(k)$ can be seen as a morphism of quiver representations. We write

$$\mathbf{M}_{\mathrm{c}}(k) \longrightarrow \mathbf{M}(k) \tag{9}$$

for the respective linear hulls, and call *canonical* the functor (9). In terms of algebras and modules,

Theorem 3.6. *There exists a quiver representation* $\lambda : \mathrm{Q} \longrightarrow \mathcal{D}^{\mathrm{b}}(\mathbf{M}_{\mathrm{c}})$ *such that the diagram*

$$\begin{tikzcd} \mathrm{Q}_{\mathrm{c}} \arrow[r, "\text{can.lift}"] \arrow[d, "\subseteq"'] & \mathbf{M}_{\mathrm{c}} \arrow[d, "\text{can.}"] \\ \mathrm{Q} \arrow[r, "\text{can.lift}"'] \arrow[ur, "H^0\circ\lambda"] & \mathbf{M} \end{tikzcd}$$

commutes up to natural isomorphisms. Moreover, equalities $\lambda([X,Y,n,i]) = \lambda([X,Y,0,0])[-n](i)$ *hold, and for all triples* $[X,Y,Z]$ *the triangles*

$$\lambda([X,Y,n,i]) \longrightarrow \lambda([X,Z,n,i]) \longrightarrow \lambda([Y,Z,n,i]) \longrightarrow \lambda([X,Y,n+1,i])$$

are exact, where morphisms are the images under λ *of the corresponding morphisms of type (a) for inclusions and of type (b) for the triple.*

The construction of λ uses two essential ingredients: One is the cell decomposition theorem 3.3 which we use to define a complex for every object $[X,Y,n,i]$ where X is affine, and the other is Jouanoulou's trick, which permits us to replace a general variety by an affine one which is homotopic to it. Having done so, we obtain a complex in $\mathbf{M}_{\mathrm{c}}$ which is our candidate for $\lambda([X,Y,n,i])$, but depends on several choices. Once we look at the complex as an object in the derived category $\mathcal{D}^{\mathrm{b}}(\mathbf{M}_{\mathrm{c}})$, we are rid of all dependence on these choices.

Corollary 3.7. *The canonical functor (9) is an equivalence of categories.*

Proof. This follows from 3.6 and the general equivalence criterion 1.14. □

Corollary 3.8. *The category of motives* $\mathbf{M}(k)_R$ *is equivalent to the category of finitely generated* R*-modules equipped with a continuous* E*-module structure for an* R*-proalgebra* $E = \lim_i E_i$*, where each* R*-algebra* E_i *is finitely generated and projective as an* R*-module.*

Proof. The category $\mathbf{M}_{\mathrm{c}}(k)_R$ has this property by definition. Indeed, for every finite subquiver Q of $\mathrm{Q}_{\mathrm{c}}(k)$, the R-module $\mathrm{End}(\rho_s|_Q)$ is projective because for every object $q \in \mathrm{Q}_c$ the R-module $\rho(q)$, and hence the R-module $\mathrm{End}(\rho(q))$ is projective. □

Corollary 3.9. *Every motive* M *over* K *is a subquotient of a sum of motives of the form* $H^n([X,Y])(i)$*, where* $[X,Y]$ *is of the form*

$$X = \overline{X} \setminus Y_\infty \qquad\qquad Y = Y_0 \setminus (Y_0 \cap Y_\infty)$$

for some smooth projective, connected variety $\overline{X}$ *of dimension* n *and effective divisors* Y_0 *and* Y_∞ *on* $\overline{X}$ *such that* $Y_0 \cup Y_\infty$ *is a strict normal crossings divisor.*

Proof. By Corollary 1.12 and Theorem 3.6 we know that every motive M over K is a subquotient of a sum of motives of the form $H^n([X,Y])(i)$ for some quadruple $[X,Y,n,i]$, where X is affine of dimension n, and $Y \subseteq X$ of dimension $\leq n-1$. It thus suffices to prove the statement of the corollary for the motive $M = H^n([X,Y])$. Let $\widetilde{Y} \subseteq X$ be a subvariety of dimension $\leq n-1$ which contains Y and the singular locus of X. The morphism $H^n([X,\widetilde{Y}]) \longrightarrow H^n([X,Y])$ is surjective, hence we may replace Y by $\widetilde{Y}$ and thus suppose that the complement of Y in X is smooth. Finally, we choose a compactification of X, and use resolution of singularities to produce $\overline{X}$, Y_0 and Y_∞. Notice that in a situation

$$\begin{array}{ccccc} X' \setminus Y' = U' & \longrightarrow & X' & \longleftarrow & Y' \\ \cong\downarrow & & f_X\downarrow & & f_Y\downarrow \\ X \setminus Y = U & \longrightarrow & X & \longleftarrow & Y \end{array} \qquad \dim X' = n, \quad \dim Y' \leq n-1$$

the morphism of pairs (f_X, f_Y) induces an isomorphism $H^n([X,Y]) \longrightarrow H^n([X',Y'])$ (for the corresponding statement for homology, see [Hat02], Proposition 2.22). □

- 3.10. Let me recall the following observation, due to Jouanolou ([Jou73], Lemme 1.5): For every quasiprojective variety X over k, there exists an affine variety $\widetilde{X}$ and a morphism $p : \widetilde{X} \longrightarrow X$ such that each fibre $p^{-1}(x)$ is isomorphic to $\mathbb{A}^d$ for some $d \geq 0$ (but there is no such thing as a zero-section $X \longrightarrow \widetilde{X}$). In particular, the induced continuous map $\widetilde{X}(\mathbb{C}) \longrightarrow X(\mathbb{C})$ is a homotopy equivalence! The proof is simple: For $X = \mathbb{P}^n$ take for $\widetilde{X}$ the variety of $(n+1)\times(n+1)$ matrices of rank 1 up to scalars with its obvious map to $\mathbb{P}^n$, and for general X choose a projective embedding and do a pullback. Let us call such a morphism $p : \widetilde{X} \longrightarrow X$ an *affine homotopy replacement.*

Jouanoulou's trick does not give a functorial homotopy replacement of varieties X by affine $\widetilde{X}$, but nearly so. Given a morphism of varieties $Y \longrightarrow X$, we can replace first X by an affine $\widetilde{X} \longrightarrow X$, and then Y by an affine homotopy replacement $\widetilde{Y}$ of the fibre product $Y \times_X \widetilde{X}$. The map $\widetilde{Y} \longrightarrow Y$ is an affine homotopy replacement, and we obtain a morphism $\widetilde{Y} \longrightarrow \widetilde{X}$ which lifts the given morphism $Y \longrightarrow X$. This procedure can be generalised to the case of several morphisms from $Y \longrightarrow X$, but not to arbitrary diagrams of varieties.

Definition 3.11. Let X be an affine variety over k and let $Z \subseteq Y \subseteq X$ be a closed subvarieties. A *cellular filtration* of $[X, Y, Z]$ is a chain of closed immersions

$$\varnothing \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X_{p-1} \subseteq X_p \subseteq \cdots \subseteq X_d = X \tag{10}$$

where each X_p is of dimension $\leq p$, such that the pairs

$$[X_p, X_{p-1}], \quad [Y_p, Y_{p-1}], \quad [Z_p, Z_{p-1}], \quad [X_p, Y_p \cup X_{p-1}], \quad [Y_p, Z_p \cup Y_{p-1}]$$

are cellular in degree p, for $Y_p := X_p \cap Y$ and $Z_p := X_p \cap Z$. By a cellular filtration of $[X, Y]$ we understand a cellular filtration of $[X, Y, \varnothing]$.

Proposition 3.12. *Let X be an affine variety over k and let $Z \subseteq Y \subseteq X$ be closed subvarieties. There exist cellular filtrations of $[X, Y, Z]$, and every filtration of X by closed subvarieties X_p of dimension $\leq p$ is contained in a cellular filtration.*

Proof. This is a direct consequence of the cell decomposition theorem 3.3: Suppose we are given a filtration of the form (10), which satisfies the cellularity condition for $j \geq p+1$. By 3.3 there exists a closed subvariety Z of dimension $\leq p-1$ of X_p such that the pairs

$$[X_p, X_{p-1} \cup Z], \quad [Y_p, Y_{p-1} \cup (Y_p \cap Z)], \quad [Z_p, Z_{p-1} \cup (Z_p \cap Z)]$$
$$[X_p, (Y_p \cup X_{p-1}) \cup Z], \quad [Y_p, (Z_p \cup Y_{p-1}) \cup (Z \cap Y_p)]$$

are cellular in degree p. Replace then X_{p-1} by $X_{p-1} \cup Z$ and continue by induction on p. □

- 3.13. Let X be an affine variety over k and $Y \subseteq X$ be a closed subvariety, let X_* be a cellular filtration of $[X,Y]$ and set $Y_p = X_p \cap Y$. We will consider the complexes

$$C^*(X_*, Y_*) = \big[\cdots \longrightarrow H^p([X_p, Y_p \cup X_{p-1}]) \xrightarrow{d_p} H^{p+1}([X_{p+1}, Y_{p+1} \cup X_p]) \longrightarrow \cdots\big] \tag{11}$$

in the category $\mathbf{M}_c$, where the differential d_p is the connecting morphism in the long exact sequence associated with the triple $X_{p-1} \subseteq X_p \subseteq X_{p+1}$ and the sheaf on X which computes the cohomology of the pair $[X,Y]$. Let me clarify this. For every constructible sheaf F on X and every triple $X_{p-1} \subseteq X_p \subseteq X_{p+1}$ there is a long exact sequence

$$\cdots \longrightarrow H^n([X_{p+1}, X_{p-1}], F) \longrightarrow H^n([X_p, X_{p-1}], F) \xrightarrow{\partial} H^{n+1}([X_{p+1}, X_p], F) \longrightarrow \cdots \tag{12}$$

of R-modules. Consider this sequence for the three sheaves in the standard short exact sequence $0 \longrightarrow \beta_!\beta^*\underline{R}_X \longrightarrow \underline{R}_X \longrightarrow \alpha_*\alpha^*\underline{R}_X \longrightarrow 0$ of sheaves on X, where α is the inclusion of Y into X, and β the inclusion of the open complement. We have

$$\begin{aligned} H^n([X_p, X_{p-1}], \beta_!\beta^*\underline{R}_X) &= H^n([X_p, Y_p \cup X_{p-1}]) \\ H^n([X_p, X_{p-1}], \alpha_*\alpha^*\underline{R}_X) &= H^n([Y_p, Y_{p-1}]) \end{aligned}$$

by definition of the cohomology of a relative pair, hence (12) and the cellularity assumptions yield a morphism of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & H^p([X_p, Y_p \cup X_{p-1}]) \arrow[r] \arrow[d, "d_p"'] & H^p([X_p, X_{p-1}]) \arrow[r] \arrow[d, "\partial"'] & H^p([Y_p, Y_{p-1}]) \arrow[r] \arrow[d, "\partial"'] & 0 \\ 0 \arrow[r] & H^{p+1}([X_{p+1}, Y_{p+1} \cup X_p]) \arrow[r] & H^{p+1}([X_{p+1}, X_p]) \arrow[r] & H^{p+1}([Y_{p+1}, Y_p]) \arrow[r] & 0 \end{tikzcd} \tag{13}$$

in which the differential of (11) appears. This diagram shows as well that d_p is a morphism in $\mathbf{M}_c$ rather than just a morphism of modules, indeed, all other morphisms in the diagram are morphisms in $\mathbf{M}_c$ since they either are given by inclusions of pairs or by connecting morphisms of triples, and hence come from morphisms in Q_c. That the composite $d_{p-1} \circ d_p$ is zero follows from the fact that for any chain of closed subvarieties $X_{p-2} \subseteq X_{p-1} \subseteq X_p \subseteq X_{p+1}$ of X and any sheaf F on X, the composite

$$H^{p-1}([X_{p-1}, X_{p-2}], F) \longrightarrow H^p([X_p, X_{p-1}], F) \longrightarrow H^{p+1}([X_{p+1}, X_p], F)$$

is zero. The complex $C^*(X_*, Y_*)$ is functorial in the obvious way for for morphisms of filtered pairs: Let $f : X' \longrightarrow X$ be a morphism of affine varieties over k, restricting to a morphism $Y' \longrightarrow Y$ between closed subvarieties, and let X_* and X'_* be cellular filtrations for $[X,Y]$ and $[X',Y']$ such that $f(X'_p)$ is contained in X_p and $f(Y'_p)$ in Y_p for all $p \geq 0$. The morphism

$$C^*(f) : C^*(X_*, Y_*) \longrightarrow C^*(X'_*, Y'_*) \tag{14}$$

shall be the one induced by the morphism $H^p([X_p, Y_p \cup X_{p-1}]) \longrightarrow H^p(X'_p, Y'_p \cup X'_{p-1})$ given by the restriction of f to X'_p.

Lemma 3.14. *Let X be a variety over k and let $Y \subseteq X$ be a closed subvariety. Let $\widetilde{X}_*$ be a cellular filtration of $[\widetilde{X}, \widetilde{Y}]$ for some affine homotopy replacement $\widetilde{X} \longrightarrow X$. There is a natural isomorphism*

$$C^*([\widetilde{X}_*, \widetilde{Y}_*]) \cong \mathbb{R}\pi_*(\beta_!\beta^*\underline{R}_X) \tag{15}$$

in the derived category of R-modules, where $\pi : X \longrightarrow \operatorname{spec} k$ is the structural morphism of X and β the inclusion of the complement of Y into X.

Proof. Let me clarify what the two complexes are. On the right hand side stands the derived functor $\mathbb{R}\pi_*$ applied to the sheaf $\beta_!\beta^*\underline{R}_X$ - it is calculated by choosing an injective resolution of the sheaf $\beta_!\beta^*\underline{R}_X$ and applying to this resolution the global sections functor π_*. On the left hand side we have a complex of motives, which has an underlying complex of modules. It is given in degree p by the module $H^p([\widetilde{X}_p, \widetilde{X}_p \cap \widetilde{Y}], R)$. Thus, the claim of the Lemma is the following:

Claim: Let $\pi : X \longrightarrow S$ be a continuous map of topological spaces, let F be a sheaf on X, and let X_* be a finite exhaustive filtration of X by closed subspaces X_p such that $H^n([X_p, X_{p-1}], F)$ is zero for $n \neq p$ (relative cohomology of X over S). Then the complex of sheaves

$$\cdots \longrightarrow H^{p-1}([X_{p-1}, X_{p-2}], F) \longrightarrow H^p([X_p, X_{p-1}], F) \longrightarrow H^{p+1}([X_{p+1}, X_p], F) \longrightarrow \cdots \tag{16}$$

on S is isomorphic to $\mathbb{R}\pi_*(F)$ in the derived category of sheaves on S.

To see this, choose an injective resolution $F \longrightarrow I_*$ of F. The long exact sequence (16) is natural in F, so if we apply it to I_* we obtain a double complex, hence a spectral sequence with initial terms $H^q([X_{p-1}, X_{p-2}], F)$ convering to $H^{p+q}(X, F)$. This spectral sequence degenerates by the assumption on the filtration, and yields the desired quasiisomorphism of complexes.

Naturality of the isomorphism (15) for morphisms of filtered pairs follows from naturality of (16) in X_* and F. □

Proposition 3.15. *Let X be an affine variety over k, let $Y \subseteq X$ be closed subvariety and let X_* be a cellular filtration of $[X, Y]$. There is a canonical isomorphism in $\mathbf{M}$*

$$H^p(C^*([X_*, Y_*]) \cong H^p([X, Y])$$

which is natural for morphisms of filtered pairs. If X is of dimension $\leq n$ and $[X, Y]$ cellular in degree n, then this isomorphism is an isomorphism in $\mathbf{M}_{\mathrm{c}}$.

Proof. The cohomology of $C^*(X_*, Y_*)$ in degree p is the object

$$H^p(C^*(X_*, Y_*)) = \frac{\ker\big(H^p([X_p, Y_p \cup X_{p-1}]) \longrightarrow H^{p+1}([X_{p+1}, Y_{p+1} \cup X_p])\big)}{\operatorname{im}\big(H^{p-1}([X_{p-1}, Y_{p-1} \cup X_{p-2}]) \longrightarrow H^p([X_p, Y_p \cup X_{p-1}])\big)} \tag{17}$$

in $\mathbf{M}_{\mathrm{c}}$ and we wish to show that this object is naturally isomorphic to $H^p(X, Y)$ in $\mathbf{M}$, and even in $\mathbf{M}_{\mathrm{c}}$ if $[X, Y]$ is cellular. To treat cases uniformly, pick any finite subquiver Q of $\mathrm{Q}(k)$ or of $\mathrm{Q}_{\mathrm{c}}(k)$ which contains and least $[X, Y, p, 0]$, the $[X_p, Y_p \cup X_{p-1}, p, 0]$, the morphisms coming from inclusions, and the connecting morphisms of triples, subject to future enlargment. Set $E = \operatorname{End}(\rho_s|_Q)$. For all integers $q < p$ and n, we have $H^n(X_p, Y_p \cup X_q) = 0$ unless $q < n \leq p$. Indeed, this is true by definition if $q = p - 1$, and follows in general by induction on $p - q$ using the long exact sequence associated with the triple $X_q \subseteq X_{p-1} \subseteq X_p$. This explains why the morphisms

$$H^p(X, Y) \longrightarrow H^p(X_{p+1}, Y_{p+1}) \longleftarrow H^p(X_{p+1}, Y_{p+1} \cup X_{p-2}) \tag{18}$$

are isomorphisms of R-modules, and also explains the surjections and injections in the following diagram, whose exact rows and columns are pieces of the long exact sequences associated with triples out of the quadruple $X_{p-2} \subseteq \cdots \subseteq X_{p+1}$.

$$\begin{tikzcd}
H^{p-1}(X_{p-1}, Y_{p-1} \cup X_{p-2}) \arrow[r, "\partial"] \arrow[d, equal, "*"'] & H^p(X_{p+1}, Y_{p+1} \cup X_{p-1}) \arrow[r, two heads] \arrow[d, hook] & H^p(X_{p+1}, Y_{p+1} \cup X_{p-2}) \arrow[d, hook] \\
H^{p-1}(X_{p-1}, Y_{p-1} \cup X_{p-2}) \arrow[r, "\partial", "*"'] & H^p(X_p, Y_p \cup X_{p-1}) \arrow[r, two heads] \arrow[d, "*"', "\partial"] & H^p(X_p, Y_p \cup X_{p-2}) \arrow[d, "\partial"'] \\
& H^{p+1}(X_{p+1}, Y_{p+1} \cup X_p) \arrow[r, equal, "*"'] & H^{p+1}(X_{p+1}, Y_{p+1} \cup X_p)
\end{tikzcd}$$

This diagram is a diagram of R-modules where all morphisms labelled with a $*$ are morphisms of E-modules between E-modules. But then the whole diagram is a diagram of E-modules, in only one possible way. Now we have an E-module structure on $H^p([X, Y])$ and on $H^p(X_{p+1}, Y_{p+1} \cup X_{p-2})$, and we need to show that the isomorphisms (18) are isomorphisms of E-modules after possibly enlarging Q. In the case where we work with subquivers of $\mathrm{Q}(k)$ we add to Q the two morphisms of pairs needed to define (18) and are done. If we work with cellular pairs only, then X has dimension $\leq p$ and $[X, Y]$ is cellular in degree p, and we enlarge Q as follows: By 3.3, there exists a closed $Z \subseteq X$ of dimension $\leq p-1$ such that $H^p(X, Y')$ is cellular in degree p for $Y' := Y \cup X_{p-1} \cup Z$. Add the morphism $[X, Y', n, 0] \longrightarrow [X, Y, n, 0]$ to Q so that $H^p([X, Y']) \longrightarrow H^p([X, Y])$ is an E-linear morphism. It is surjective for dimension reasons, and the diagram of E-modules and R-linear maps

$$\begin{tikzcd}
H^p([X, Y']) \arrow[r, two heads, "*"] \arrow[d, "*"'] & H^p([X, Y]) \arrow[d, "\cong"', "u"] \\
H^p([X_p, Y_p \cup X_{p-1}]) \arrow[r, two heads, "*"'] & H^p([X_p, Y_p \cup X_{p-2}])
\end{tikzcd}$$

commutes, where the isomorphism of R-modules u is induced by (18). All morphisms labelled $*$ are E-linear hence so is u. Altogether, we conclude that the homology in the middle of

$$H^{p-1}([X_{p-1}, Y_{p-1} \cup X_{p-2}]) \longrightarrow H^p([X_p, Y_p \cup X_{p-1}]) \longrightarrow H^{p+1}([X_{p+1}, Y_{p+1} \cup X_p])$$

is indeed canonically isomorphic to $H^p(X, Y)$ as an E-module, which is what we had to show. Naturality of the isomorphism for morphisms of filtered pairs follows from functoriality of (18). $\square$

Corollary 3.16. *Let $f : X' \longrightarrow X$ be a morphism of affine varieties restricting to a morphism of closed subvarieties $Y' \longrightarrow Y$. Let X_* and X'_* be compatible cellular filtrations of $[X, Y]$ and $[X', Y']$. If the morphism of relative topological spaces $[X'(\mathbb{C}), Y'(\mathbb{C})] \longrightarrow [X(\mathbb{C}), Y(\mathbb{C})]$ is a homotopy equivalence, then the morphism of complexes $C^*([X_*, Y_*]) \longrightarrow C^*([X'_*, Y'_*])$ defined in (14) is a quasiisomorphism.*

Proof. This follows from statement (4) of Proposition 2.5 and Proposition 3.15. $\square$

Proposition 3.17. *Let X be an affine variety over k, let $Z \subseteq Y \subseteq X$ and $Z \subseteq Y$ be closed subvarieties and let X_* be a cellular filtration of $[X, Y, Z]$. The sequence of complexes with morphisms given by (14) for inclusions*

$$0 \longrightarrow C^*([X_*, Y_*]) \longrightarrow C^*([X_*, Z_*]) \longrightarrow C^*([Y_*, Z_*]) \longrightarrow 0 \tag{19}$$

is degreewise exact.

Proof. The sequences in question are sequences in $\mathbf{M}_c$, but in order to show that they are exact it suffices to show that the underlying sequence in $\mathbf{Modf}_R$ are exact. But that immediately follows from the definition of cellular filtrations and a diagram chase. □

Proof of Theorem 3.6. For every object $[X, Y, n, i]$ of $\mathrm{Q}(k)$ and cellular filtration X_* of $[X, Y]$, we consider the complex $C^*(X_*, Y_*)[-n](i)$ obtained from $C^*([X_*, Y_*])$ by shifting and twisting degree by degree. Let us define λ by

$$\lambda([X, Y, n, i]) = \operatorname*{colim}_{\widetilde{X} \to X} \lim_{\widetilde{X}_*} C^*(\widetilde{X}_*, \widetilde{Y}_*)[-n](i)$$

on objects $[X, Y, n, i]$ of $\mathrm{Q}(k)$, where the limit runs over all cellular filtrations of the pair $[\widetilde{X}, \widetilde{Y}]$ and the colimit over all affine homotopy replacements $\pi : \widetilde{X} \longrightarrow X$, setting $\widetilde{Y} := \widetilde{X} \times_X Y$. These colimits and limits exist in the derived category $\mathcal{D}^{\mathrm{b}}(\mathbf{M}_c)$, indeed, all transition maps are isomorphisms by Corollary 3.16. From the practical point of view, $\lambda([X, Y, n, i])$ is isomorphic to any of the complexes $C^*(\widetilde{X}_*, \widetilde{Y}_*)[-n](i)$ up to a unique isomorphism in $\mathcal{D}^{\mathrm{b}}(\mathbf{M}_c)$, and the use of the limiting processes is only to get rid of choices[4]. We define λ on morphisms as follows:

Type (a): Let $f : [X, Y, n, i] \longrightarrow [X', Y', n, i]$ be given by a morphism of varieties $f : X' \longrightarrow X$ restricting to $Y' \longrightarrow Y$. From (14) we obtain a morphism $C^*(f) : C^*([\widetilde{X}'_*, \widetilde{Y}'_*]) \longrightarrow C^*([\widetilde{X}_*, \widetilde{Y}_*])$ for suitable affine homotopy replacements and cellular filtrations, and set $\lambda(f) := C^*(f)[-n](i)$.

Type (b): Let $d : [Y, Z, n, i] \longrightarrow [X, Y, n + 1, i]$ be given by closed immersions $Z \subseteq Y \subseteq X$ between affine varieties. Choose an affine homotopy replacement $\widetilde{X} \longrightarrow X$, set $\widetilde{Y} = \widetilde{X} \times_X Y$ and $\widetilde{Z} = \widetilde{X} \times_X Z$ and cellular filtration of the triple $[\widetilde{X}, \widetilde{Y}, \widetilde{Z}]$. From Proposition 3.17 we obtain an degreewise exact sequence of complexes

$$0 \longrightarrow C^*([\widetilde{X}_*, \widetilde{Y}_*]) \xrightarrow{r} C^*([\widetilde{X}_*, \widetilde{Z}_*]) \xrightarrow{s} C^*([\widetilde{Y}_*, \widetilde{Z}_*] \longrightarrow 0 \tag{20}$$

where r and s induced by inclusions, hence a morphism in $\mathcal{D}^{\mathrm{b}}(\mathbf{M}_c)$ given by the hat

$$\begin{tikzcd} & \mathrm{Cone}(r) \arrow[dl, "\simeq \text{ induced by } s"'] \arrow[dr] & \\ C^*([Y, Z]) \arrow[rr, dash, "\partial"] & & C^*([X, Y])[-1] \end{tikzcd}$$

and define $\lambda(d) = \partial[-n](i)$.

Type (c): If $\widetilde{X} \longrightarrow X$ is an affine homotopy replacement, then so is $\widetilde{X} \times \mathbb{G}_m \longrightarrow X \times \mathbb{G}_m$. If $\widetilde{X}_*$ a cellular filtration of $[\widetilde{X}, \widetilde{Y}]$, then the $\widetilde{X}_p \times \mathbb{G}_m \subseteq \widetilde{X} \times \mathbb{G}_m$ form a cellular filtration of $[\widetilde{X} \times \mathbb{G}_m, \widetilde{Y} \times \mathbb{G}_m \cup \widetilde{X} \times \{1\}]$. Hence there is a canonical isomorphism of complexes

$$C^*([\widetilde{X}_* \times \mathbb{G}_m, \widetilde{Y}_* \times \mathbb{G}_m \cup \widetilde{X}_* \times \{1\}])(1) \longrightarrow C^*([\widetilde{X}_*, \widetilde{Y}_*]) \tag{21}$$

obtained from the corresponding isomorphisms degree-by-degree, and we declare this morphism shifted and twisted by $[-n](i)$ to be the image under λ of the morphism of type (c) with target $[X, Y, n, i]$.

Now that we have defined λ, it remains to show that the diagram in the statement of Theorem 3.6 indeed commutes up to natural isomorphisms. All other statement hold by construction. The

[4] provided one disposes of a concrete construction of limits in $\mathcal{D}^{\mathrm{b}}(\mathbf{M}_c)$.

isomorphisms we seek

$$\lambda([X, Y, n, i]) \cong H^n([X, Y])(i)$$

are those of Proposition 3.15 with a twist. Naturality of these isomorphisms for morphisms in $\mathrm{Q}(k)$ is a question on the level of modules, and follows from the fact that the isomorphisms in Proposition 3.15 are induced, as morphisms of modules, by the isomorphisms of complexes of 3.14. □

4. Motives of simplicial varieties

Let U and V be open subvarieties of a variety X, such that X is the union of U and V. The cohomology groups of X, U, V and $U \cap V$ are related by the *Maier-Vietoris* sequence:

$$\cdots \longrightarrow H^n(X) \longrightarrow H^n(U) \oplus H^n(V) \longrightarrow H^n(U \cap V) \xrightarrow{\partial} H^{n+1}(X) \longrightarrow \cdots$$

All terms in this sequence are motives, and differentials which connect cohomology groups of the same degree are morphisms of motives, since they are induced by inclusions of varieties. Also the connecting morphism ∂ is a morphism of motives, but this is not clear à priori, and we shall prove it in this section.

- 4.1. We keep the assumptions of 3.1, and write $\mathbf{M}$ in place of $\mathbf{M}(k)_R$.

- 4.2. A sheaf on a simplicial topological space $X_\bullet$ is the data of a sheaf F_n on each X_n and compatibility morphisms for faces and cofaces. One can regard sheaves $F_\bullet$ on $X_\bullet$ as sheaves on some convenient site, and using that point of view, the cohomology $H^n(X_\bullet, F_\bullet)$ is defined. There is a spectral sequence of groups

$$E_1^{p,q} = H^q(X_p, F_p) \implies H^{p+q}(X_\bullet, F_\bullet) \tag{22}$$

constructed as follows: An injective resolution of $F_\bullet$ gives an injective resolution of each F_n. Applying global sections gives a simplicial complex of R-modules, hence a double complex. Its horizontal differentials (say) are alternating sums of face maps, and its vertical differentials are induced by the differentials of the resolution. The spectral sequence (22) is the one associated with this double complex.

Let $f : X_0 \longrightarrow X$ be a continuous map between topological spaces. We can regard f as a morphism from $X_\bullet$ to the constant simplicial space X. There is a pair of adjoint functors (f^*, f_*) between the categories of sheaves on $X_\bullet$ and on X. The map f is said to have the property of *cohomological descent* if the adjunction transform

$$F \longrightarrow \mathbb{R}f_*(f^*F)$$

is an isomorphism in the derived category of the category of sheaves on X, for every sheaf or complex of sheaves F on X. In that case, there is an isomorphism $H^n(X_\bullet, f^*F) \longrightarrow H^n(X, F)$ for every sheaf of R-modules F on X, and the spectral sequence (22) translates to a spectral sequence

$$E_1^{p,q} = H^q(X_p, f^*F_p) \implies H^{p+q}(X, F) \tag{23}$$

where the induced filtration on the cohomology groups $H^n(X,F)$ is the one that comes from truncations of $X_\bullet$ (the q-skelettons of $X_\bullet$ for $q = 0,1,2,\ldots$).

- 4.3. Let $X_\bullet$ be an affine simplicial variety over k and let $Y_\bullet$ be a closed simplicial subvariety of $X_\bullet$. We wish to equip the R-modules $H^n([X_\bullet, Y_\bullet])$ with a natural motivic structure. We can do it as follows: Successively choose for each X_n an affine replacement $\widetilde{X}_n$ and lifts of face maps δ_i^n to obtain a diagram of varieties

$$\widetilde{X}_\bullet := \Big[\cdots \qquad \widetilde{X}_3 \underset{\widetilde{\sigma}_0^2,\widetilde{\sigma}_1^2,\widetilde{\sigma}_2^2}{\overset{\widetilde{\delta}_0^3,\widetilde{\delta}_1^3,\widetilde{\delta}_2^3,\widetilde{\delta}_3^3}{\rightleftarrows}} \widetilde{X}_2 \underset{\widetilde{\sigma}_0^1,\widetilde{\sigma}_1^1}{\overset{\widetilde{\delta}_0^2,\widetilde{\delta}_1^2,\widetilde{\delta}_2^2}{\rightleftarrows}} \widetilde{X}_1 \underset{\widetilde{\sigma}_0^0}{\overset{\widetilde{\delta}_0^1,\widetilde{\delta}_1^1}{\rightleftarrows}} \widetilde{X}_0 \Big]$$

where the solid arrows $\widetilde{\delta}_i^n$ are morphisms of algebraic varieties lifting the face maps δ_i^n, and where the dashed arrows indicate morphisms $X_i \longrightarrow X_{i+1}$. We cannot lift the σ_i^n to morphisms $\widetilde{X}_i \longrightarrow \widetilde{X}_{i+1}$, so $\widetilde{X}_\bullet(\mathbb{C})$ is a not a simplicial topological space, but it is a simplicial object in the category of topological space localised in homotopy equivalences. Setting $\widetilde{Y}_n := \widetilde{X}_n \times_X Y$, we obtain a similar diagram $\widetilde{Y}_\bullet$. Next, for some big integer $N \gg 0$, choose cellular filtrations $X_{n,*}$ of $[\widetilde{X}_n, \widetilde{Y}_n]$ for $n = N, N-1, \ldots, 0$ such that the $\widetilde{\delta}_i^n$ respect these filtrations. This is possible by Proposition 3.12. We obtain a double complex

$$\begin{array}{ccccccc}
\cdots & & \cdots & & \cdots & & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
H^2([X_{0,2}, Y_{0,2}]) & \to & H^2([X_{1,2}, Y_{1,2}]) & \longrightarrow & \cdots & \longrightarrow & H^2([X_{N,2}, Y_{N,2}]) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
H^1([X_{0,1}, Y_{0,1}]) & \to & H^1([X_{1,1}, Y_{1,1}]) & \longrightarrow & \cdots & \longrightarrow & H^1([X_{N,1}, Y_{N,1}]) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
H^0([X_{0,0}, Y_{0,0}]) & \to & H^0([X_{1,0}, Y_{1,0}]) & \longrightarrow & \cdots & \longrightarrow & H^0([X_{N,0}, Y_{N,0}])
\end{array}$$

in $\mathbf{M}$. Let $C^*([X_\bullet, Y_\bullet])_N$ be the associated simple complex. Up to a canonical isomorphism in $\mathcal{D}^{\mathrm{b}}(\mathbf{M})$, it does not depend on the choice of the homotopy replacements and cellular filtrations. We have a canonical morphisms $C^*([X_\bullet, Y_\bullet])_{N+1} \longrightarrow C^*([X_\bullet, Y_\bullet])_N$ in $\mathcal{D}^{\mathrm{b}}(\mathbf{M})$ which induce isomorphisms $H^n(C^*([X_\bullet, Y_\bullet])_{N+1}) \longrightarrow H^n(C^*([X_\bullet, Y_\bullet])_N)$ for $n < N$. We can define a motivic structure on $H^n([X_\bullet, Y_\bullet])$ by setting

$$H^n([X_\bullet, Y_\bullet]) := H^n(C^*([X_\bullet, Y_\bullet])_N)$$

for any $N > n$. With this structure, morphisms of simplicial relative varieties $[X'_\bullet, Y'_\bullet] \longrightarrow [X_\bullet, Y_\bullet]$ induce morphisms of motives $H^n([X_\bullet, Y_\bullet]) \longrightarrow H^n([X'_\bullet, Y'_\bullet])$, and if $[X_\bullet, Y_\bullet]$ is a constant simplicial object defined by $[X, Y]$, we have $H^n([X_\bullet, Y_\bullet]) = H^n([X, Y])$.

Theorem 4.4. *Let $[X_\bullet, Y_\bullet] \longrightarrow [X, Y]$ be an augmented simplicial object in the category of relative varieties over k. There is a spectral sequence*

$$E_1^{p,q} = H^q([X_p, Y_p]) \qquad \Longrightarrow \qquad H^{p+q}([X_\bullet, Y_\bullet])$$

in $\mathbf{M}$, whose underlying spectral sequence of R-modules is that of a hypercover given in (23). The augmentation induces morphisms $H^n([X_\bullet, Y_\bullet]) \longrightarrow H^n([X, Y])$ in $\mathbf{M}$.

Proof. The spectral sequence whose existence we claim is the one associated with the double complex displayed in 4.3 in the region where the indices p, q satisfy $p+q < N$. As we let N grow, this eventually defines the whole spectral sequence. That its underlying spectral sequence of R-modules is the one given in (23) follows from Lemma 3.14. $\square$

Corollary 4.5. *Let X be a variety over k, let $Y \subseteq X$ be a closed subvariety, and let U and V be open subvarieties of X satisfying $U \cup V = X$, and set $W := U \cap V$. All morphisms in the Maier-Vietoris sequence*

$$\cdots \longrightarrow H^p([U, U\cap Y]) \oplus H^p([V, V\cap Y]) \longrightarrow H^p([W, W\cap Y]) \xrightarrow{\partial} H^{p+1}([X,Y]) \longrightarrow \cdots$$

are morphisms of motives.

Proof. With the open cover $f : U \sqcup V \longrightarrow X$ is associated a hypercovering of X, the coskeleton of f. The Maier-Vietoris sequence appears in the corresponding spectral sequence. $\square$

5. Cohomology with support and the Gysin map

In this section we introduce a motivic structure on cohomology with support on a closed subvariety, and on cohomology with compact support. Then we show that the projective bundle formula and Gysin morphisms are morphisms of motives.

- 5.1. As in the previous section, k is a subfield of $\mathbb{C}$, and R is a commutative, coherent ring. All motives are motives over k with coefficients in R.

- 5.2. We begin with defining a motivic structure on cohomology with support. Let X be a variety over k, and let Y and Z be closed subvarieties of X. Write U for the open complement of Z and set $V := U \cap Y$. There is a long exact sequence

$$\cdots \longrightarrow H^n_Z([X,Y]) \longrightarrow H^n([X,Y]) \longrightarrow H^n([U,V]) \longrightarrow H^{n+1}_Z([X,Y]) \longrightarrow \cdots \tag{24}$$

of R-modules, and we wish to equip $H^n_Z([X,Y])$ with a motivic structure such that the sequence becomes a sequence of motives. To this end, choose an affine homotopy replacements $\widetilde{X}$ and $\widetilde{U} \subseteq \widetilde{X}$ of X and U (Jouanolou's trick, c.f. §3.10), set $\widetilde{Y} = \widetilde{X} \times_Y$ and $\widetilde{V} = \widetilde{U} \times_U V$, and choose compatible cellular filtrations $\widetilde{X}_*$ of $[\widetilde{X}, \widetilde{Y}]$ and $\widetilde{U}_*$ of $[\widetilde{U}, \widetilde{V}]$. We obtain a morphism of complexes of motives $C^*([\widetilde{X}_*, \widetilde{Y}_*]) \longrightarrow C^*([\widetilde{U}_*, \widetilde{V}_*])$ from (14). Set

$$C^*_Z([X,Y]) := \mathrm{Cone}\big(C^*([\widetilde{X}_*, \widetilde{Y}_*]) \longrightarrow C^*([\widetilde{U}_*, \widetilde{V}_*])\big)[1]$$

and define the motivic structure on $H^n_Z([X,Y])$ by transport via the natural isomorphism of R-modules

$$H^n_Z([X,Y]) = H^n(C^*_Z([X,Y]))$$

given by Lemma 3.14. This structure does not depend on the choices made, is natural in X, Y and Z, and is compatible with the differentials in (24).

- 5.3. Let us introduce a motivic structure on cohomology with compact support. Let X be a variety over k, and let Y be a closed subvariety of X. We define

$$H^n_c([X,Y]) := H^n([\overline{X}, Y \cup Z])$$

where $\overline{X}$ is any compactification of X, and Z the complement of X in $\overline{X}$. The so defined motive $H^n_c([X,Y])$ is independent of the choice of the compactification $\overline{X}$, and fits into the usual long exact sequence relating cohomology and cohomology with compact support.

Proposition 5.4. *Set $\mathbb{1}(i) := H^0(\operatorname{spec} k)(i)$. There are canonical isomorphisms of motives:*

(1) *$H^n(\mathbb{P}^d_k) = \mathbb{1}(-i)$ if $n = 2i \leq 2d$ is even, and $H^n(\mathbb{P}^d_k) = 0$ if n is odd or $n > 2d$.*
(2) *$H^0(\mathbb{A}^d_k \setminus \{0\}) = \mathbb{1}$, $H^{2d-1}(\mathbb{A}^d_k \setminus \{0\}) = \mathbb{1}(-d)$ and $H^n(\mathbb{A}^d_k \setminus \{0\}) = 0$ for $n \notin \{0, 2d-1\}$.*
(3) *$H^{2d}(X) = \mathbb{1}(-d)$ for every projective and geometrically connected variety X of dimension d over k.*

Proof. The morphisms of type (c) in the standard quiver representation 1.4 induce isomorphisms $H^d(\mathbb{G}^d_m)(0) \cong H^{d-1}(\mathbb{G}^{d-1}_m)(-1) \cong \cdots \cong H^0(\operatorname{spec} k)(-d) = \mathbb{1}(-d)$. From the standard covering of the projective space $\mathbb{P}^d_k$ by $d+1$ affine spaces we obtain a spectral sequence (the Maier-Vietoris sequence if $d = 1$), in which the connecting morphism

$$H^d(\mathbb{G}^d_m) \longrightarrow H^{2d}(\mathbb{P}^d_k)$$

appears. This connecting morphism is an isomorphism of modules because $\mathbb{A}^d_k$ is contractible as well as a morphism of motives by Theorem 4.4 (or Corollary 4.5 for $d = 1$), hence it is an isomorphism of motives. The first statement of the proposition follows from this isomorphism, induction on d and the fact that the inclusion of a hyperplane $\mathbb{P}^{d-1}_k \longrightarrow \mathbb{P}^d_k$ induces an isomorphism on cohomology in degrees up to $2d-2$. The argument for the second statement is similar - one covers $\mathbb{A}^d_k \setminus \{0\}$ by the affine open $\mathbb{A}^a \times \mathbb{G}_m \times \mathbb{A}^b$ with $a + b = d - 1$. To show the third statement of the proposition, choose a projective embedding $f : X \longrightarrow \mathbb{P}^N_k$. The morphism f induces a morphism of motives $\mathbb{1}(-d) \cong H^{2d}(\mathbb{P}^N_k) \longrightarrow H^{2d}(X)$ which is an isomorphism of modules, hence an isomorphism of motives. $\square$

Proposition 5.5. *Suppose that R is a field or a Dedekind ring. Let X be a smooth variety over k, and let $E \longrightarrow X$ be a vector bundle of rank r over X, with projectivisation $\mathbb{P}(E) \longrightarrow X$. Denote by 0_X the image of the zero section $X \longrightarrow E$. There are isomorphisms in $\mathbf{M}(k)$ as follows:*

$$H^n(\mathbb{P}(E)) \cong \bigoplus_{i=0}^{r-1} H^{n-2i}(X)(i) \qquad \textit{and} \qquad H^*(E \setminus 0_X) \cong H^n(X) \oplus H^{n-2r+1}(X)(r)$$

Proof. Proposition 5.4 and the Künneth formula settle the case where E is constant. The Künneth isomorphism is indeed a morphisms of motives by Theorem 6.2. We have not yet proven this theorem, but the reader can check that there is no circular argument. The hypothesis on R is needed in Theorem 6.2. For the general case, choose a finite covering of X by open subvarieties $(U_i)_{i\in I}$ on which E is isomorphic to the constant bundle of rank r. With the open covering $(\mathbb{P}(E) \times_X U_i)_{i\in I}$ of $\mathbb{P}(E)$ is ssociated a simplicial variety, which in turn yields a double complex of motives. The motive $H^*(\mathbb{P}(E))$ is the homology of the associated simple complex as we have explained in 4.3. The same way we can compute the motive of $\mathbb{P}(\mathbb{A}^r_k) \times_k X$. For every finite intersection V of the U_i's, choose an isomorphism of bundles $\alpha_V : \mathbb{A}^r_k \times_k V \cong E \times_X V$. Because $\mathrm{GL}_r(\mathbb{C})$ is connected, these isomorphisms induce isomorphisms of motives

$$H^n(\mathbb{P}(E) \times_X V) \cong H^n(\mathbb{P}(\mathbb{A}^r_k) \times_k V)$$

which are independent of the choice of α_V. It follows that these isomorphisms commute with the differentials in the double complexes, hence induce an isomorphism of motives. The same argument works for the sphere bundle $E \setminus 0_X \longrightarrow X$. □

Proposition 5.6. *Let X be a smooth, irreducible variety over k, and let $Z \subseteq X$ be a smooth and irreducible subvariety of codimension c, with open complement U. There is a long exact sequence of motives*

$$\cdots \longrightarrow H^n(X) \longrightarrow H^n(U) \longrightarrow H^{n-2c+1}(Z)(-c) \longrightarrow H^{n+1}(X) \longrightarrow \cdots$$

whose underlying sequence of modules is the usual Gysin sequence.

Proof. We use deformation to the normal cone - see [Ful98] Chapter 5 for an exposition. Let $\widetilde{X}$ be the blowup of $Z \times \{0\}$ in $X \times \mathbb{A}^1_k$, and $\pi : \widetilde{X} \longrightarrow \mathbb{A}^1$ be the composite of the blowup map $\widetilde{X} \longrightarrow X \times \mathbb{A}^1_k$ and the projection $X \times \mathbb{A}^1_k \longrightarrow \mathbb{A}^1_k$. The fibre of π over any nonzero point of $\mathbb{A}^1_k$ is equal to X. The fibre $\pi^{-1}(0)$ has two irreducible components. One component is $\mathbb{P}(N_Z \oplus \mathcal{O}_Z)$, the projective completion of the normal bundle $N_X := T_X Z/T_Z Z$ of Z in X. The other component is $\mathrm{Bl}_Z X$, the blowup of Z in X. These two components intersect in $\mathbb{P}(N_X)$, seen as infinity section in $\mathbb{P}(N_Z \oplus \mathcal{O}_Z)$ and as the exceptional divisor in $\mathrm{Bl}_Z X$.

The inclusions $\pi^{-1}(0) \longrightarrow \widetilde{X}$ and $X \cong \pi^{-1}(1) \longrightarrow \widetilde{X}$ induce morphisms of motives as follows:

$$H^n([\pi^{-1}(0), \mathrm{Bl}_Z X]) \xleftarrow{(*)} H^n([\widetilde{X}, \mathrm{Bl}_Z X]) \longrightarrow H^n(X)$$

The map labelled $(*)$ is an isomorphism (of modules, hence of motives) and we have canonical isomorphisms $H^n([\pi^{-1}(0), \mathrm{Bl}_Z X]) \cong H^n\mathbb{P}([N_Z \oplus \mathcal{O}_Z), \mathbb{P}(N_Z)]) \cong H^{n-2c}(Z)(c)$ induced by inclusion and the projective bundle formula 5.5. We obtain a morphism of motives

$$H^{n-2c}(Z)(c) \longrightarrow H^n(X) \tag{25}$$

whose underlying morphism of modules is the Gysin map (that is, the proper pushforward map for the inclusion $Z \longrightarrow X$).

The rest of the argument is formal. The projective bundle formula can be seen as an isomorphism

$$C^*(\mathbb{P}(E)) \cong \bigoplus_{i=0}^{r-1} C^*(X)[2i](i)$$

in the derived category $\mathcal{D}^{\mathrm{b}}(\mathbf{M})$. The morphism of pairs $[\pi^{-1}(0), \mathrm{Bl}_Z X] \longrightarrow [\widetilde{X}, \mathrm{Bl}_Z X]$ induces therefore a morphism $C^*(Z)[2c](c) \longrightarrow C^*(X)$ in $\mathcal{D}^{\mathrm{b}}(\mathbf{M})$ inducing (25). Its composition with $C^*(X) \longrightarrow C^*(U)$ is zero, hence a morphism

$$C^*(Z)[2c](c) \longrightarrow C^*_Z(X)$$

in $\mathcal{D}^{\mathrm{b}}(\mathbf{M})$. This morphism is indeed an isomorphism, because the underlying morphism in the derived category of R-modules is so. The Gysin sequence in the statement of the proposition is, via this isomorphism, the sequence of cohomology with suport (24). □

Corollary 5.7. *Let X be a smooth variety over k, and let Y and Z be closed subvarieties of $\overline{X}$ such that $Y+Z$ is a strict normal crossings divisor on X. There is a natural isomorphism of motives $H^n_Z([X,Y]) \cong H^n([Z, Y\cap Z])(1)$.*

Proof. If Z and Y have only one component, this follows by comparing the Gysin sequences for the pairs $Z \subseteq X$ and $Y \cap Z \subseteq Y$ with the long exact sequence for cohomology with support on X and on Y. The general case follows by induction on the number of components. □

Corollary 5.8. *Every object M of $\mathbf{M}(k)$ admits a finite and exhaustive filtration, such that the associated graded object $\mathrm{gr}^*(M)$ is isomorphic to a subquotient of a sum of objects of the form $H^n(X)(i)$, where X is a smooth projective variety.*

Proof. Every motive M is a subquotient of a sum of motives $H^n([X,Y])(i)$ for some $[X,Y,n,i]$, and $H^n([X,Y])(i)$ sits in the long exact sequence of the triple $[X,Y,\varnothing]$. It suffices thus to prove the statement of the proposition for $M = H^n(X)$. Using resolution of singularities we may construct a hypercovering of X by smooth varieties, hence by Theorem 4.4 it suffices to prove the statement of the proposition for $M = H^n(X)$ where X is smooth. Again using resolution of singularities, there is a smooth compactification $\overline{X}$ of X by a strict normal crossings divisor Z. We can further reduce to the case where $Z \subseteq X$ is a smooth subvariety, and conclude using Proposition 5.6. □

6. Construction of a monoidal structure

In this section, we construct a monoidal structure on the category of motives $\mathbf{M}$, and show that realisation functors are monoidal. In the case where we work with coefficients in a field, $\mathbf{M}$ will become a tannakian category.

- 6.1. We fix a field $k \subseteq \mathbb{C}$, and a ring of coefficients R. As a standing technical hypothesis (in order to apply Proposition 1.18), we suppose throughout this section that R is a Dedekind ring or a field. Also, we regard $\mathbf{M} = \mathbf{M}(k)_R$ throughout the section as the linear hull of the quiver representation

$$\rho : \mathrm{Q_c}(k) \longrightarrow \mathbf{Modf}_R$$

where $\mathrm{Q_c} = \mathrm{Q_c}(k)$ is the quiver of cellular relative varieties defined in 3.5. This point of view is justified by Corollary 3.7.

Theorem 6.2. *The category $\mathbf{M}$ admits a unique R-linear monoidal structure that satisfies the following properties.*

1. *The forgetful functor $\mathrm{R_B} : \mathbf{M} \longrightarrow \mathbf{Modf}_R$ is strictly monoidal.*
2. *Künneth morphisms are morphisms of motives.*

With this monoidal structure, $\mathbf{M}$ is a symmetric monoidal closed R-linear category. If R is a field, then $\mathbf{M}$ is a neutral tannakian category, with the forgetful functor as fibre functor.

Let me spell out in more detail what Theorem 6.2 states and indicate the steps of the proof. A symmetric monoidal structure on $\mathbf{M}$ is a functor $\otimes : \mathbf{M} \times \mathbf{M} \longrightarrow \mathbf{M}$ which we call *tensor product* together with isomorphisms of functors expressing associativity and commutativity of the tensor product, and the properties of $\mathbb{1}(0) = H^0(\operatorname{spec} k)$ playing the role of a neutral object. We construct the functor $\otimes$ in 6.3, associativity and unit constraints in 6.4 and the commutativity constraint in 6.5. That the forgetful functor or Betti realisation $\mathrm{R_B} : \mathbf{M} \longrightarrow \mathbf{Modf}_R$ is strictly monoidal means that there exist natural isomorphisms

$$\mathrm{R_B}(M \otimes N) \cong \mathrm{R_B}(M) \otimes \mathrm{R_B}(N) \tag{26}$$

that are compatible with the associativity and commutativity constraints. These isomorphisms will come directly from the construction of the tensor product in 6.3, and will be equalities. Given relative varieties $[X, Y]$ and $[X', Y']$, the Künneth morphisms are morphisms of modules

$$H^n([X,Y]) \otimes H^{n'}([X',Y']) \longrightarrow H^{n+n'}([X \times X', (Y \times X') \cup (X \times Y')])$$

and property (2) states that these morphisms are compatible with the motivic structures. That $\mathbf{M}$ is monoidally closed means that the functor $- \otimes M$ has a natural right adjoint, which we denote by $\underline{\mathrm{Hom}}(M, -)$, so that the usual adjunction formula

$$\mathrm{Hom}(X \otimes M, Y) \cong \mathrm{Hom}(X, \underline{\mathrm{Hom}}(M, Y))$$

holds.

- 6.3 (Construction of the tensor product)**.** We construct a functor $\otimes : \mathbf{M} \times \mathbf{M} \longrightarrow \mathbf{M}$, to be called *tensor product*, using the following morphism of quiver representations.

$$\begin{tikzcd} \mathrm{Q_c} \boxtimes \mathrm{Q_c} \arrow[rr, "\mathrm{prod}"] \arrow[dr, "\rho\otimes\rho"'] & & \mathrm{Q_c} \arrow[dl, "\rho"] \\ & \mathbf{Modf}_R & \end{tikzcd} \tag{27}$$

The quiver $\mathrm{Q_c} \boxtimes \mathrm{Q_c}$ is defined in 1.16, and the horizontal quiver morphism is given by

$$([X, Y, n, i], [X', Y', n', i']) \longmapsto [X \times X', (X \times Y') \cup (Y \times X'), n + n', i + i']$$

on objects, and the obvious map on morphisms. The Künneth formula provides us with the required natural isomorphism

$$H^{n+n'}([X \times X', (X \times Y') \cup (Y \times X')])(i + i') \cong H^n([X, Y])(i) \otimes H^{n'}([X', Y'])(i') \tag{28}$$

so that the diagram (27) together with (28) constitutes a morphism of quiver representations. By Proposition 1.18, this morphism of quiver representations induces an exact and faithful functor $\mathbf{M} \otimes \mathbf{M} \longrightarrow \mathbf{M}$ which corresponds, by the universal property of the tensor product category, to a functor

$$\otimes : \mathbf{M} \times \mathbf{M} \longrightarrow \mathbf{M}$$

which is right exact in both variables. Concretely, we compute their tensor product of motives $M = (V, Q, \alpha)$ and $M' = (V', Q', \alpha')$ as follows. We start by replacing the finite quivers Q and Q' by sufficiently large finite quivers consisting of cellular objects. Let $Q \times Q'$ be the quiver consisting of objects $[X \times X', (X \times Y') \cup (Y \times X'), n+n', i+i']$ where $[X, Y, n, i]$ is an object of Q and $[X', Y', n', i']$

an object of Q', and morphisms either of the form $(f \times \mathrm{id}'_X)$ or $(\mathrm{id} \times f')$ for a morphism f in Q or a morphism f' in Q'. We set

$$M \otimes M' = (V \otimes V', Q \times Q', \beta)$$

where the action β of $\mathrm{End}(\rho_s|_{Q\times Q'})$ on $V \otimes V'$ is the composite

$$\mathrm{End}(\rho_s|_{Q'\times Q}) \underset{(*)}{\cong} \mathrm{End}(\rho_s \otimes \rho_s|_{Q\boxtimes Q'}) \underset{(**)}{\cong} \mathrm{End}(\rho_s|_Q) \otimes \mathrm{End}(\rho_s|_Q) \xrightarrow{\alpha\otimes\alpha'} \mathrm{End}(V \otimes V')$$

where the isomorphism $(*)$ is induced by the Künneth formula (28) and $(**)$ is given by Proposition 1.18. This explicit description shows that the forgetful functor is indeed strictly monoidal, in fact, the natural isomorphism (26) is an actual equality $\mathrm{R_B}(M \otimes M') = \mathrm{R_B}(M) \otimes \mathrm{R_B}(M')$.

- 6.4 (Construction of the associativity and unit constraint)**.** The associativity constraint is a natural isomorphism $M \otimes (M' \otimes M'') \cong (M \otimes M') \otimes M''$. We obtain it using the the general construction 1.9 as follows: We have a morphisms of quiver representations

$$\begin{tikzcd} \mathrm{Q_c} \boxtimes \mathrm{Q_c} \boxtimes \mathrm{Q_c} \arrow[rr, shift left, "\mathrm{prod}\circ(\mathrm{prod}\boxtimes\mathrm{id})"] \arrow[rr, shift right, "\mathrm{prod}\circ(\mathrm{id}\boxtimes\mathrm{prod})"'] \arrow[dr, "\rho^{\otimes 3}"'] & & \mathrm{Q_c} \arrow[dl, "\rho"] \\ & \mathbf{Modf}_R & \end{tikzcd}$$

$$s : \rho \circ \mathrm{prod} \circ (\mathrm{prod} \boxtimes \mathrm{id}) \longrightarrow \rho^{\otimes 3}$$

$$t : \rho \circ \mathrm{prod} \circ (\mathrm{id} \boxtimes \mathrm{prod}) \longrightarrow \rho^{\otimes 3}$$

where s and t are given by the Künneth isomorphism (28). A 2-morphism between these quiver morphisms is given by the morphisms of motives

$$H^{n+n'+n''}([X \times (X' \times X''), \ldots]) \cong H^{n+n'+n''}([(X \times X') \times X'', \ldots])$$

obtained from the isomorphism of varieties $X \times (X' \times X'') \cong (X \times X') \times X''$ - to ease the notation I did not write out the closed subvarieties and the Tate twists. The functor isomorphism induced by this 2-morphism is the sought commutativity constraint. The unit constraint is constructed similarly, using the isomorphisms of varieties $X \times \mathrm{spec}(k) \cong X \cong \mathrm{spec}(k) \times X$ and the identification $H^0(\mathrm{spec}\, k) = R$. The forgetful functor is strictly compatible with these constraints.

- 6.5 (Construction of the commutativity constraint)**.** We have to construct an isomorphism of functors between the tensor product functor, and the tensor product functor precomposed with the switch-of-factors functor. We use again the general construction 1.9 to do this. Consider the morphisms of quiver representations (27) and its switched variant

$$\begin{tikzcd} \mathrm{Q_c} \boxtimes \mathrm{Q_c} \arrow[rr, shift left, "\mathrm{prod}"] \arrow[rr, shift right, "\mathrm{prod}\circ\mathrm{sw}"'] \arrow[dr, "\rho\otimes\rho"'] & & \mathrm{Q_c} \arrow[dl, "\rho"] \\ & \mathbf{Modf}_R & \end{tikzcd}$$

$$s : \rho \circ \mathrm{prod} \longrightarrow \rho \otimes \rho$$

$$t : \rho \circ \mathrm{prod} \circ \mathrm{sw} \longrightarrow \rho \otimes \rho$$

where the natural transform s is the Künneth isomorphism, and t the Künneth isomorphism composed with the switch of R-modules, as displayed in the diagram (29) below. For every object $q = ([X, Y, n, i], [X', Y', n', i'])$ of $\mathrm{Q_c} \boxtimes \mathrm{Q_c}$, let

$$\widetilde{\eta}_q : [X \times X', X \times Y' \cup Y \times X', n + n', i + i'] \longrightarrow [X' \times X, X' \times Y \cup Y' \times X, n' + n, i' + i]$$

be the morphism of type (a) in $\mathrm{Q_c}$ induced by the morphism of varieties $X \times X' \longrightarrow X' \times X$, and set $\eta_q = (-1)^{nn'}\rho(\widetilde{\eta}_q)$. This sign in η_q renders the diagram

$$\begin{tikzcd}
H^{n'+n}([X',Y'] \times [X,Y])(i'+i) \arrow[r, "\text{Kü}"] \arrow[d, "\eta_q"'] \arrow[dr, "t_q"] & H^{n'}([X',Y'])(i') \otimes_R H^n([X,Y])(i) \arrow[d, "\mathrm{sw}"] \\
H^{n+n'}([X,Y] \times [X',Y'])(i+i') \arrow[r, "\text{Kü}", "s_q"'] & H^n([X,Y])(i) \otimes_R H^{n'}([X',Y'])(i')
\end{tikzcd} \tag{29}$$

commutative, and the η_q are morphisms in $\mathbf{M}$, hence constitute a 2-morphism as in 1.9. The induced isomorphism of functors is the sought commutativity constraint. It is strictly compatible with the commutativity constraint in the category of R-modules.

- 6.6. So far, we have constructed an R-linear symmetric monoidal structure on $\mathbf{M}$ which satisfies the two properties in the statement of Theorem 6.2. The unicity statement is evident: Any other monodical structure having the required properties must agree with the one we constructed on objects in $\mathrm{Q_c} \boxtimes \mathrm{Q_c}$, hence on all of $\mathbf{M} \otimes \mathbf{M}$. It remains show that this monoidal structure is closed, or in other words, that there exist internal homomorphisms in $\mathbf{M}$.

Let A be a cocommutative bialgebra in the procategory of the category of finitely generated projective R-modules, and denote by $\mathbf{A}$ the category of continuous left A-modules which are finitely presented as R-modules. The category $\mathbf{A}$ is a symmetric monoidal category with unit R. Let M and N be A-modules. We will say that an A-linear map

$$\beta : M \otimes_R N \longrightarrow R$$

is a *nondegenerate duality* if the two induced morphisms of R-modules $M \longrightarrow \mathrm{Hom}_R(N,R)$ and $N \longrightarrow \mathrm{Hom}_R(M,R)$ are injective. If these induced morphisms are even bijective, we say that β is a *perfect duality.* By a *nondegenerate* or *perfect coduality* we understand an A-linear map

$$\gamma : R \longrightarrow M \otimes_R N$$

such that the induced morphisms of R-modules $\mathrm{Hom}_R(M,R) \longrightarrow N$ and $\mathrm{Hom}_R(N,R) \longrightarrow M$ are injective or bijective respectively. Lemma 6.7 is a standard fact from the theory of tannakian formalism, at least if R is a field (see for example [Sz09], Theorem 6.3.4).

Lemma 6.7. *Suppose that $\mathbf{A}$ is generated, as an R-linear abelian monoidal category, by objects M which admit a nondegenerate coduality $R \longrightarrow M \otimes_R N$. Then, $\mathbf{A}$ is monoidal closed and A is a Hopf algebra.*

Proof. The monoidal category $\mathbf{A}$ is closed if and only if A is a Hopf algebra, so we have to show that there exists an antipode $\iota : A \longrightarrow A$. Let us write $A = \lim_i A_i$ where all A_i are R-algebras which are finitely generated and projective as R-modules. If an object of $\mathbf{A}$ admits a nondegenerate codual, all its submodules and all its quotients do as well. Any sum or tensor product of objects that admit a nondegenerate codual admit a nondegenerate codual. Hence, all objects of $\mathbf{A}$ admit a nondegenerate coduality, and in particular each A_i.

Let M be an A-module M which is finitely generated and projective as an R-module. I claim that if M admits a nondegenerate coduality $\gamma : R \longrightarrow M \otimes N$, then M admits a perfect duality.

Indeed, that M admits a nondegenerate duality can be shown by a standard computation over the field of fractions of R and multiplying by a common denominator the obtained pairing. Here we use that M is projective, hence torsion free. Now let $\beta : M \otimes_R N' \longrightarrow R$ be any nondegenerate duality. The image of β is an ideal I of R, and there exists an ideal J of R such that $I \otimes J \cong R$. By modifying the pairing β by

$$M \otimes_R (N' \otimes_R J) \longrightarrow I \otimes_R J \longrightarrow R$$

we see that there is an R-module $N'' = N' \otimes_R J$ and a *perfect* pairing $M \otimes N'' \longrightarrow R$.

In summary, we can write $A = \lim_i A_i$, where each A_i is projective as an R-module and admits a perfect duality, say $A_i \otimes B_i \longrightarrow R$. The construction of ι is standard from here: Denote by A_i^0 the A-module whose underlying R-module is A_i, but with the A-module structure given by the counit $A \longrightarrow R$ and R-linearity. We then have an A-module structure on $\mathrm{Hom}_R(A_i, A_i^0) \cong B_i \otimes_R A_i^0$, say

$$\alpha_i : A \otimes \mathrm{Hom}_R(A_i, A_i^0) \longrightarrow \mathrm{Hom}_R(A_i, A_i^0)$$

and we set $\iota(a) = (\alpha_i(a \otimes 1)(1))_{i \in I} \in \lim_i A_i = A$. The verification that this $\iota : A \longrightarrow A$ indeed satisfies all properties required of an antipode is straightforward. $\square$

Proposition 6.8. *The symmetric monoidal category* $\mathbf{M}(k)_R$ *is closed.*

Proof. According to Corollary 3.8 and Lemma 6.7, it suffices to show that $\mathbf{M} = \mathbf{M}(k)_R$ contains a class of objects which admit a nondegenerate coduality and which generate $\mathbf{M}$ as an R-linear abelian monoidal category. According to Corollary 3.9, a class of objects that do generate $\mathbf{M}$ even as an abelian category are those of the form $H^{n-p}([X,Y])(i)$, where $[X,Y]$ is of the form

$$X = \overline{X} \setminus Y_\infty \qquad\qquad Y = Y_0 \setminus (Y_0 \cap Y_\infty)$$

for some smooth projective, connected variety $\overline{X}$ of dimension n and divisors Y_0 and Y_∞ on $\overline{X}$ such that $Y_0 \cup Y_\infty$ is a strict normal crossings divisor. Let us show that these objects admit coduals. Concretely, we define

$$X' = \overline{X} \setminus Y_0 \qquad\qquad Y' = Y_\infty \setminus (Y_\infty \cap Y_0)$$

and construct a coduality pairing

$$\mathbb{1}(-n) \longrightarrow H^{n-p}([X,Y]) \otimes H^{n+p}([X',Y']) \tag{30}$$

that is, a nondegenerate coduality of R-modules which is also a morphism of motives. The Tate twisting is inessential. The coduality (30) we seek to construct corresponds to the Poincaré-Verdier duality pairing.

Let Δ be the diagonal embedding of $X \cap X'$ in $X \times X'$. We declare (30) to be the following composite of morphisms of motives:

$$\begin{array}{ccc} \mathbb{1}(-n) & \xrightarrow[\cong]{(1)} & H^{2n}_\Delta([X \times X', X \times Y' \cup Y \times X']) \\ & & \big\downarrow{\scriptstyle (2)} \\ H^{n-p}([X,Y]) \otimes H^{n+p}([X',Y']) & \xleftarrow{(3)} & H^{2n}([X \times X', X \times Y' \cup Y \times X']) \end{array}$$

The isomorphism (1) is given by the Gysin morphism, noting that Δ is connected, of codimension n in $X \times X'$ and does not meet $X \times Y' \cup Y \times X'$. The morphism (2) is functoriality of cohomology with supports. The morphism (3) is any morphism which splits the Künneth morphism in the opposite direction up to multiplication with a nonzero scalar in R.

It remains to be shown that this morphism is a nondegenerate coduality. This is a question of R-modules and R-linear maps, and we can replace R by its fraction field. In that case, there are no torsion terms in the Künneth formula, and (3) exists canonically. In this situation, the non degeneracy is well known in the case where Y_0 and Y_∞ are empty, hence $X = \overline{X}$ is smooth and proper, since one obtains the Poincaré duality pairing. The general case follows by dévissage and induction on the number of components of Y_0 and Y_∞. □

7. Motivic Galois groups

We can now introduce the main protagonist of the present paper: The motivic Galois group of the field k. In the case where the coefficient ring R is a field, this group is the Galois group of the tannakian category $\mathbf{M}(k)_R$ with respect to the fibre functor $\mathrm{R_B} : \mathbf{M}(k)_R \longrightarrow \mathbf{Modf}_R$.

- 7.1. We keep the assumptions of 6.1. By definition, the category $\mathbf{M} = \mathbf{M}(k)_R$ is the category of finitely generated R-modules with a continuous action by the proalgebra $E = \mathrm{End}(\rho)$. Tensor products and duals in $\mathbf{M}(k)_R$ are given by a comultiplication and an antipode on E, which give E the structure of a cocommutative Hopf algebra object in the category of profinitely generated R-modules.

- 7.2. Let us write $E = \lim_{i\in I} E_i$ where each E_i is a an R-algebra which is finitely generated and projective as an R-module (cf. Corollary 3.8), and set $A_i = \mathrm{Hom}_R(E_i, R)$. Then each A_i is a coalgebra which is finitely generated and projective as an R-module, and hence $A := \mathrm{colim}_{i\in I} A_i$ is a commutative Hopf-algebra over R, flat as an R-module. The group scheme

$$G_{\mathrm{mot}}(k)_R := \mathrm{spec}(A)$$

is a flat group scheme over R.

Definition 7.3. We call $G_{\mathrm{mot}}(k)_R$ the *motivic Galois group* of the field k.

- 7.4. Supose that the coefficient ring $R = \mathbb{F}$ is a field. Then, to give an E_i-module structure on a finite dimensional $\mathbb{F}$-vector space M is the same as to give an A_i-comodule structure on the dual space $\mathrm{Hom}_{\mathbb{F}}(M, \mathbb{F})$. Therefore, the opposite category $\mathbf{M}(k)_{\mathbb{F}}^{\mathrm{op}}$ is equivalent to the category of finitely dimensional vector spaces together with an A-comodule structure, or in other words, the category $\mathbf{M}(k)_{\mathbb{F}}$ itself is equivalent to the category of finite dimensional $\mathbb{F}$-linear representations of the motivic Galois group $G_{\mathrm{mot}}(k)_{\mathbb{F}}$. It follows that $G_{\mathrm{mot}}(k)_{\mathbb{F}}$ is the Tannakian Galois group of the category $\mathbf{M}(k)_{\mathbb{F}}$ with respect to the forgetful functor as fibre functor.

- 7.5. Caveat: The hypothesis that R is a field is crucial in 7.4. Consider any ring E which is finitely generated and free as a $\mathbb{Z}$-module, and the corresponding coring $A := \mathrm{Hom}(E, \mathbb{Z})$. The category of E-modules is not antiequivalent to the category of A-comodules. Indeed, already for $E = \mathbb{Z}$ there is a problem: the category of finitely generated commutative groups is not equivalent to its opposite category (though their derived categories are equivalent!).

- 7.6. For every morphism between Dedekind domains $R \longrightarrow S$, there is a canonical morphism of group schemes $G_{\mathrm{mot}}(k)_R \times_R S \longrightarrow G_{\mathrm{mot}}(k)_S$ over S, induced by (7). It is in general not an isomorphism. It is however an isomorphism in the important case where S is a localisation of R, or where S is projective as an R-module, and thus in particular when $R \longrightarrow S$ is a field extension or the inclusion $\mathbb{Z} \longrightarrow \mathbb{Q}$.

Corollary 7.7 (to Proposition 1.20). *Let $R \longrightarrow S$ be a morphism between commutative rings, where each is a Dedekind domain or a field. If S is projective as an R-module, or a localisation or completion of R, then the canonical morphism $G_{\mathrm{mot}}(k)_R \times_R S \longrightarrow G_{\mathrm{mot}}(k)_S$ is an isomorphism.*

8. Filtration by weight and semisimplicity

In this section we show that motives over k with rational coefficients admit two filtrations, one by weight which maps to the weight filtration of mixed Hodge structures under the Hodge realisation functor, and the other by level. A positive answer to the Hodge conjecture would imply that hat the level filtration of motives is compatible with the level filtration of Hodge structures. We show that motives of pure weight are semisimple objects, and that the category of semisimple Nori-motives is equivalent to André's category of motives with respect to motivated correspondences. These statements were already made by Arapura ([Ara13], in particular Theorem 6.4.1), but in a different setup of which I do not know whether it is well related to ours.

- 8.1. In this section k is a subfield of $\mathbb{C}$, and $\mathbf{M}(k) := \mathbf{M}(k)_{\mathbb{Q}}$ denotes the category of mixed motives over k with rational coefficients. We denote by $\mathbf{A}(k)$ André's $\mathbb{Q}$-linear abelian category of semisimple motives over k with respect to motivated correspondences (see [And96] and section 9.2 in [And04]), as reviewed in 8.8.

Theorem 8.2. *On every motive M over k with rational coefficients, there exists a unique finite and exhaustive filtration*

$$\cdots \subseteq W_{-1}M \subseteq W_0M \subseteq W_1M \subseteq W_2M \subseteq \cdots$$

which maps to the weight filtration under the Hodge realisation functor. We call it weight filtration*. Moreover, the following holds.*

(1) *Every motive M which is of pure weight p admits a* polarisation*, that means, a morphism of motives $\psi : M \otimes M^\vee \longrightarrow \mathbb{1}$ whose underlying bilinear form of rational vector spaces is positive definite. In particular, pure motives are semisimple.*
(2) *There is a canonical functor $\iota : \mathbf{A}(k) \longrightarrow \mathbf{M}(k)$ which sends the motive in $\mathbf{A}(k)$ of a smooth and proper variety to the motive of the same variety in $\mathbf{M}(k)$ and is compatible with Betti realisations. This functor induces an equivalence between $\mathbf{A}(k)$ and the full subcategory of $\mathbf{M}(k)$ whose objects are the semisimple objects.*

Proof of Theorem 8.2 without statements (1) and (2). Let $\overline{X}$ be a smooth proper variety over k, and let Y_0 and Y_1 be smooth, closed subvarieties of $\overline{X}$ such that $Y_0 + Y_1$ is a strict normal crossings divisor. Set

$$X := \overline{X} \setminus Y_1 \qquad \text{and} \qquad Z := Y_0 \cap Y_1 \qquad \text{and} \qquad Y := Y_0 \setminus Z$$

and let us construct the weight filtration for the motive $M = H^n([X, Y])$. There is an exact diagram of motives

$$\begin{tikzcd}
H^{n-1}([Y_0, \varnothing]) \arrow[r] \arrow[d] & H^n([\overline{X}, Y_0]) \arrow[r] \arrow[d] & H^n([\overline{X}, \varnothing]) \arrow[d] \\
H^{n-1}([Y, \varnothing]) \arrow[r] \arrow[d] & H^n([X, Y]) \arrow[r] \arrow[d] & H^n([X, \varnothing]) \arrow[d] \\
H^{n-2}([Z, \varnothing])(1) \arrow[r] & H^{n-1}([Y_1, Z])(1) \arrow[r] & H^{n-1}([Y_1, \varnothing])(1)
\end{tikzcd}$$

where rows are from long exact sequences of triples, and columns from Gysin sequences. All four corners are pure, of weight $-n+1$, $-n$ or $-n-1$. The weight filtration is an exact functor on Hodge structures, so it follows that $M = H^n([X, Y])$ has these weights, and that the nontrivial steps in the weight filtration of M are given by

$$W_{-n-1}M = \operatorname{im}(H^{n-1}([Y_0, \varnothing]) \longrightarrow M) \qquad \text{and} \qquad W_{-n}M = \ker(M \longrightarrow H^{n-1}([Y_1, \varnothing])(1))$$

hence are motivic. Next, let us construct the weight filtration on motives of the form $M = H^n([X, Y])$ for arbitrary X and $Y \subseteq X$. Choose a hypercovering $(X_\bullet, Y_\bullet)$ of $[X, Y]$ where each $[X_n, Y_n]$ has the previous form. By Theorem 4.4 the spectral sequence of the hypercovering

$$E_1^{pq} : H^q([X_p, Y_p]) \implies H^{p+q}([X, Y])$$

is a spectral sequence of motives. By our previous considerations, the degree wise inclusion $W_n E_2^{pq} \longrightarrow E_2^{pq}$ is a morphism of spectral sequences of motives, hence produces a morphism of graded motives in the limit. By exactness of the weight filtration, this morphism is the inclusion $W_n \operatorname{gr} H^{p+q}([X, Y]) \longrightarrow \operatorname{gr} H^{p+q}([X, Y])$. The general case follows from this: Every motive is a subquotient of a sum of motives of the form $H^n([X, Y])(i)$, and we can conclude again by exactness of the weight filtration. □

Deligne shows in [Del74] how to equip the cohomology groups $H := H^n(X(\mathbb{C}), \mathbb{Q})$ of a complex algebraic variety X with a mixed Hodge structure. If X is smooth and projective, then this Hodge structure is polarisable and hence semisimple. For general varieties it is true, but unfortunately not stated in [Del74], that the weight graded pieces $\operatorname{gr}_p^W H$ are polarisable Hodge structures. We can deduce this well known fact (see for instance [Bei86]) as a a consequence of statement (1) in

Theorem 8.2. The proof I shall give here uses induction on the level, which is an interesting notion in its own right.

Definition 8.3. Let $l \geq 0$ be an integer. We call *motives of level* $\leq l$ the objects of the full abelian subcategory of $\mathbf{M}(k)$ generated by the objects $H^n([X,Y])(i)$ where n and i are integers, X an algebraic variety over k of dimension at most l, and $Y \subseteq X$ a closed subvariety.

- 8.4. A Hodge structure is said to be of level $\leq l$ if its nonzero Hodge numbers h^{pq} satisfy $|p-q| \leq l$. For instance, the Hodge structure $H := H^n(X(\mathbb{C}), \mathbb{Q})$ has level $\min(n, 2\dim(X)-n)$, and the Tate structures $\mathbb{Q}(i)$ have level zero for all $i \in \mathbb{Z}$. The Hodge realisation of a motive of level $\leq l$ is a Hodge structure of level $\leq l$ in this sense. To show the converse is a difficult problem.

Lemma 8.5. *Let X be a variety over k, let $Y \subseteq X$ be a closed subvariety and set $M := H^n([X,Y])$. The following holds:*

1. *The motive M is of level n.*
2. *The motives $W_{-n-1}M$ and $M/W_{-n}M$ are of level $n-1$.*

Proof. By choosing an affine homotopy replacement of X we may suppose without loss of generality that X is affine. By Theorem 3.3 there exists a cellular filtration $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_d = X$ of the pair $[X,Y]$. Set $Y_p := X_p \cap Y$. The homology in degree n of the complex

$$\cdots \longrightarrow H^{p-1}([X_{p-1}, Y_{p-1}]) \longrightarrow H^p([X_p, Y_p]) \longrightarrow H^{p+1}([X_{p+1}, Y_{p+1}]) \longrightarrow \cdots$$

is isomorphic to the motive $H^n([X,Y])$ on one hand, and is a subquotient of $H^n([X_n, Y_n])$ on the other hand. This shows part (1), because X_n is of dimension $\leq n$ by definition of a cellular filtration. Replacing X by a resolution of singularities and adding to Y exceptional divisors does not change M, hence to show statement (2) we can suppose that $[X,Y]$ is of the form

$$X := \overline{X} \setminus Y_1 \qquad \text{and} \qquad Z := Y_0 \cap Y_1 \qquad \text{and} \qquad Y := Y_0 \setminus Z$$

where $\overline{X}$ is a smooth projective variety over k, and Y_0 and Y_1 are closed subvarieties of $\overline{X}$ such that $Y_0 + Y_1$ is a strict normal crossings divisor. We consider the long exact sequence of the pair $[X,Y]$ and the Gysin sequence for the pair $[\overline{X}, Y_1]$, which we know is a sequence of motives from Corollary 5.7. By exactness of the weight filtration, the morphisms

$$W_{-n-1}H^n([X,Y]) \hookrightarrow H^{n-1}([Y_1, Z])(-1) \qquad \text{and} \qquad H^{n-1}(Y) \twoheadrightarrow \frac{H^n(X,Y)}{W_{-n}H^n(X,Y)}$$

obtained from these long sequences are injective and surjective as indicated. □

Proof of statement (1) of Theorem 8.2. Every motive is a subquotient of a sum of motives of the form $H^n([X,Y])(i)$. By exactness of the weight filtration, every motive of pure weight p is a subquotient of a sum of motives of the form $\mathrm{gr}^W_p(H^n([X,Y])(i))$, hence it suffices to prove that motives of the form $M = \mathrm{gr}^W_p H^n([X,Y])$ are polarisable. We prove this first in the special case where X is smooth and proper and Y empty. Then we prove the general case by induction on the level of M.

Let X be a smooth projective variety over k. Fix an ample line bundle on X. The Lefschetz operator $H^n(X)(1) \longrightarrow H^{n+2}(X)$ is the cup product with the class of the chosen ample line bundle, hence is a morphism of motives by Theorem 6.2. It follows that the Hodge operator on $H^*(X)$ is a morphism of motives as well. As a consequence, the canonical action of SL_2 on $H^*(X)$ (see for example §5.2 of [And04]) is motivic and primitive cycles in $H^n(X)$ form a direct factor of $H^n(X)$, hence $H^n(X)$ is polarisable.

If X is of dimension 0, then $M = \mathrm{gr}_p^W H^n([X,Y])$ is zero unless $n = p = 0$, and $H^0([X,Y])$ is polarisable. It follows that pure motives of level ≤ 0 are polarisable.

Let $n > 0$ be an integer, and suppose that every pure motive of level $< n$ is polarisable. Let X be an irreducible variety over K of dimension n and let $Y \subseteq X$ be a closed subvariety of positive codimension. It follows from Lemma 8.5 that $\mathrm{gr}_p^W H^q([X,Y])$ is polarisable, except maybe in the case $p = q = n$. But $\mathrm{gr}_n^W H^n([X,Y])$ is isomorphic to $H^n(\overline{X})$ where $\overline{X}$ is a smooth compactification of X, hence also $\mathrm{gr}_n^W H^n([X,Y])$ is polarisable. Therefore every motive of level n is polarisable. □

Corollary 8.6. *There exists an isomorphism (not at all unique or canonical) of functors*

$$\mathrm{R_B} \cong \mathrm{R_B} \circ \mathrm{gr}_*^W \tag{31}$$

which restricts to the identitiy on the full subcategory of $\mathbf{M}(k)$ *whose objects are the semisimple objects.*

Proof. Let $G_{\mathrm{mot}}^{\mathrm{red}}(k)$ be the quotient of $G_{\mathrm{mot}}(k)$ by its unipotent radical $U_{\mathrm{mot}}(k)$. The exact sequence

$$1 \longrightarrow U_{\mathrm{mot}}(k) \longrightarrow G_{\mathrm{mot}}(k) \longrightarrow G_{\mathrm{mot}}^{\mathrm{red}}(k) \longrightarrow 1 \tag{32}$$

admits a splitting: Indeed, a theorem of Mostov, see Theorem 4.3 in Chapter VIII of [Ho81], states that (in characteristic zero!) every linear algebraic group is the semidirect product of its maximal reductive quotient and its unipotent radical. That the sequence (32) splits follows formally from this result, since affine group schemes are limits of affine algebraic groups. The group $G_{\mathrm{mot}}^{\mathrm{red}}(k)$ is the tannakian Galois group of the full subcategory $\mathbf{M}_{\mathrm{ss}}(k)$ of $\mathbf{M}(k)$ whose objects are the semisimple objects. By statement (1) of Theorem 8.2, an object of $\mathbf{M}(k)$ is semisimple if and only if its weight filtration is split. Any splitting $G_{\mathrm{mot}}^{\mathrm{red}}(k) \longrightarrow G_{\mathrm{mot}}(k)$ of the exact sequence defines a functor $\mathbf{M}(k) \longrightarrow \mathbf{M}_{\mathrm{ss}}(k)$ which is necessarily isomorphic to gr_*^W by exactness of the weight filtration, *and* an isomorphism of fibre functors $\mathrm{R_B} \cong \mathrm{R_B} \circ \mathrm{gr}_*^W$ with the required propery. □

- 8.7. The weight grading on the category $\mathbf{M}_{\mathrm{ss}}(k)$ of semisimple motives corresponds to a cocharacter $\omega : \mathbb{G}_m \longrightarrow G_{\mathrm{mot}}^{\mathrm{red}}(k)$, and in order to obtain an isomorphism (31) we only need to lift that cocharacter to $G_{\mathrm{mot}}(k)$. In other words, we only need to split the sequence

$$1 \longrightarrow U_{\mathrm{mot}}(k) \longrightarrow G' \longrightarrow \mathbb{G}_m \longrightarrow 1$$

which is obtained by pulling back (32) along ω. The group G' is solvable and connected, and we can use the Levy decomposition to replace Mostov's theorem.

- 8.8. Let me briefly recall the the important properties of André's $\mathbb{Q}$-linear abelian category $\mathbf{A}(k)$ of motives over k with respect to motivated cycles. *Motivated cycles* constitute the smallest class of cycles up to homological equivalence on smooth projective varieties which contains the algebraic cycles, and is stable under direct and inverse images and the inverse Lefschetz isomorphisms $*_L$. Given a smooth projective variety X over k, it can be shown that a cycle $c \in H^{2p}(X, \mathbb{Q})$ is motivated if and only if it is of the form

$$\pi_*(a \cdot *_L(b))$$

where Y is a smooth projective variety, a and b algebraic cycles on $X \times Y$, $\pi : X \times Y \longrightarrow X$ the projection, and $*_L$ the inverse of the Lefschetz isomorphism with respect to some polarisation of $X \times Y$. Motivated cycles form a subalgebra of $H^{2*}(X, \mathbb{Q})$ ([And96], 2.1) and Künneth projectors as well as Lefschetz operators are motivated if one looks at them as cycles on $X \times X$ ([And96], 2.2). Objects in $\mathbf{A}(k)$ are triples (e, X, i), which we prefer to write as

$$M = e\mathfrak{h}(X)(i)$$

where X is a smooth projective variety, $e \in H^{2*}(X \times X, \mathbb{Q})$ a motivated cycle satisfying $e^2 = e$, and i an integer. If e is the Künneth projector which cuts out cohomology in degree n, we use the notation $\mathfrak{h}^n(X)(i)$ in place of $e\mathfrak{h}(X)(i)$. The set of morphisms from $e_1\mathfrak{h}(X_1)(i_1)$ to $e_2\mathfrak{h}(X_2)(i_2)$ is the set of motivated cycles in $e_2 \cdot H^{2(\dim X_1 - i_1 + i_2)}(X_1 \times X_2, \mathbb{Q}) \cdot e_1$. Composition of morphisms is given by the usual formula for composition of correspondences. There is a monoidal structure on $\mathbf{A}(k)$, given by the Künneth isomorphism modified by a sign (as we had to do in 6.5), turning $\mathbf{A}(k)$ into a graded, $\mathbb{Q}$-linear, semisimple Tannakian category ([And96], 4.4). There are canonical realisation functors from $\mathbf{A}(k)$ to the categories of rational Hodge structures or ℓ-adic Galois representations.

- 8.9. Motives with respect to motivated cycles are obtained from smooth projective varieties, but one can also obtain such motives from general relative varieties. Indeed, with every quadruple $[X, Y, n, i]$ one can associate an object in $\mathbf{A}(k)$. This object should be interpreted as the weight graded, hence semisimple motive that comes with the mixed motive $H^n([X, Y])(i)$. Its realisation in any realisation category is the realisation of the weight graded object associated with $H^n([X, Y])(i)$. I lack a reference for this construction, so I will summarise it in the next paragraph.

- 8.10. Let me recall the idea of a *weight complex* and its construction, which was introduced by Gillet and Soulé in [GS96] and further developed in [GS09]. The *weight complex* $h(X)$ of a variety X over k is a bounded complex in the category of Chow motives with rational coefficients, well defined up to homotopy of complexes, and up to homotopy depending functorially on X. The Hodge realisaton of $h(X)$ is a complex of semisimple Hodge structures, whose homology is naturally isomorphic to $\mathrm{gr}^W_* H^*(X(\mathbb{C}), \mathbb{Q})$. Bondarko [Bon07] extended the construction of the weight complex of varieties to an exact functor on Voevodsky's $\mathrm{DM}_{\mathrm{gm}}(k)$. In a nutshell, the construction of the weight complex works as follows: Let X be a quasiprojective variety over k of dimension $\leq n$, choose a projective compactification $\overline{X}$ of X and denote by Z the complement of

X in $\overline{X}$. There exists a simplicial varieties $Z_\bullet \longrightarrow \overline{X}_\bullet$ with augmentations such that

$$\begin{array}{ccc} Z_\bullet & \to & \overline{X}_\bullet \\ \downarrow & & \downarrow \\ Z & \longrightarrow & \overline{X} \end{array}$$

commutes, where each X_n and Z_n is a smooth and projective, and such that $\overline{X}_\bullet(\mathbb{C}) \longrightarrow \overline{X}(\mathbb{C})$ and $Z_\bullet(\mathbb{C}) \longrightarrow Z(\mathbb{C})$ are hypercoverings of topological spaces. From the simplicial varieties $Z_\bullet \longrightarrow \overline{X}_\bullet$ we obtain complexes of Chow motives and a map between these complexes. The cone of this morphism reads

$$\big[h(\overline{X}_0) \longrightarrow h(\overline{X}_1) \oplus h(Z_0) \longrightarrow h(\overline{X}_2) \oplus h(Z_1) \longrightarrow \cdots\big]$$

and is an unbounded complex of Chow motives, but it was shown in Theorem 2 of [GS96] that it is isomorphic in the homotopy category of the category of complexes of Chow motives to a complex of the form

$$h(X) = \big[h(X_0) \longrightarrow h(X_1) \longrightarrow \cdots \longrightarrow h(X_n) \longrightarrow 0 \longrightarrow \cdots\big]$$

where X_p is a smooth proper variety of dimension $\leq n-p$. We call $h(X)$ the weight complex of X. Notice that in the special case where X is smooth and projective we get back the Chow motive of X in degree zero. That the construction of the weight complex does not depend on choices was shown in [GS96]. Functoriality of $h(X)$ was shown in [GS96] for proper morphisms, and in [GS09] in general. It is no problem to extend the definition of weight complexes to pairs $[X,Y]$ of a variety X and a closed subvariety Y. Triples $Z \subseteq Y \subseteq X$ will induce exact triangles $h([X,Y]) \longrightarrow h([X,Z]) \longrightarrow h([Y,Z]) \longrightarrow h([X,Y])[-1]$, and there is an isomorphism

$$h([X,Y])(-1) \cong h([X \times \mathbb{G}_m, Y \times \mathbb{G}_m \cup X \times \{1\}])$$

coming from the decomposition of Chow motives $h(\mathbb{P}^1) = h(\operatorname{spec} k) \oplus h(\operatorname{spec} k)(-1)$ and parts (iv) and (v) of Theorem 2 of [GS96] which are statements about compatibility of the weight complex functor with the Maier-Vietoris sequence and products. Finally, for every pair of varieties $[X,Y]$, there is a natural isomorphism of Hodge structures[5]

$$\operatorname{gr}^W_*\big(H^*([X,Y],\mathbb{Q})\big) \cong H^*(\mathrm{R}_{\mathrm{Hdg}}(h([X,Y]))$$

essentially by definition of the mixed Hodge structure on $H^*([X,Y],\mathbb{Q})$. On the left hand side stands the weight graded object of this Hodge structure, and on the right hand side stands the homology of the Hodge realisation $\mathrm{R}_{\mathrm{Hdg}}(h([X,Y])$ of the weight complex.

- 8.11. Let $\mathrm{Q}_\mathrm{c}(k)$ be the full subquiver of $\mathrm{Q}(k)$ consisting of cellular objects, that is (as in 3.5), those objects $q = [X,Y,n,i]$ such that X is affine of dimension $\leq n$ and $[X,Y]$ is a cellular pair in degree n. For every object q of $\mathrm{Q}_\mathrm{c}(k)$, set

$$\lambda(q) := H^*(\mathfrak{h}([X,Y])$$

where $\mathfrak{h}([X,Y])$ is the weight complex of $[X,Y]$ as introduced in the previous paragraph, but now viewed as a complex in the abelian category $\mathbf{A}(k)$. Functoriality of the weight complex for all

[5]It is not an isomorphism of *graded* objects. Notice that there are two gradings on the left hand side, but only one grading on the right. The grading on the right which is missing would involve Künneth projectors.

morphisms in $\mathrm{Q_c}(k)$ yields a quiver representation

$$\lambda : \mathrm{Q_c}(k) \longmapsto \mathbf{A}(k) \tag{33}$$

and since we only consider cellular objects we obtain a natural isomorphism of vector spaces

$$\mathrm{R_B}\big(\mathrm{gr}^W_*(H^n([X,Y],\mathbb{Q}))\big) \cong \mathrm{R_B}\big(H^*(\mathfrak{h}([X,Y])\big)$$

where on the left hand side we regard $H^n([X,Y],\mathbb{Q})$ as an object of $\mathbf{M}(k)$.

Proposition 8.12. *There exists a faithful, exact and $\mathbb{Q}$-linear functor $\mathrm{R_A} : \mathbf{M}(k) \longrightarrow \mathbf{A}(k)$, unique up to isomorphism, such that the diagram*

$$\begin{tikzcd}
 & \mathrm{Q_c}(k) \arrow[dl, "\widetilde{\rho}"'] \arrow[dr, "H^*\circ\mathfrak{h}"] & \\
\mathbf{M}(k) \arrow[rr, "\mathrm{R_A}"] \arrow[dr, "\mathrm{R_B}\circ\mathrm{gr}^W_*"'] & & \mathbf{A}(k) \arrow[dl, "\mathrm{R_B}"] \\
 & \mathbf{Modf}_{\mathbb{Q}} &
\end{tikzcd}$$

commutes, and which has the following properties:

(1) *There is a natural isomorphism $\mathrm{R_A}(H^n(X)(i)) \cong \mathfrak{h}^n(X)(i)$ in $\mathbf{A}(k)$ for every smooth projective variety X over k.*

(2) *There is a natural isomorphism of Hodge structures $\mathrm{gr}^W_*\mathrm{R_{Hdg}}(M) \cong \mathrm{R_{Hdg}}(\mathrm{R_A}(M))$.*

Proof. That the outer square of the diagram commutes was explained in 8.11. Once we have chosen an isomorphism of functors $\mathrm{R_B} \cong \mathrm{R_B} \circ \mathrm{gr}^W_* : \mathbf{M}(k) \longrightarrow \mathbf{Modf}_{\mathbb{Q}}$, which exists by Corollary 8.6, the existence and unicity of the functor $\mathrm{R_A}$ follows from Theorem 1.10 and its consequences stated in 1.11. That two different choices of an isomorphism $\mathrm{R_B} \cong \mathrm{R_B} \circ \mathrm{gr}^W_*$ yield isomorphic functors $\mathrm{R_A}$ follows from 1.9. Let us show (1). Given a smooth projective variety X over k, we choose an affine homotopy replacement $\widetilde{X}$ of X, and a cellular filtration of $\widetilde{X}$, so we get a complex of motives

$$\cdots \longrightarrow H^p([\widetilde{X}_p, \widetilde{X}_{p-1}], \mathbb{Q}) \longrightarrow H^{p+1}([\widetilde{X}_{p+1}, \widetilde{X}_p], \mathbb{Q}) \longrightarrow \cdots$$

whose homology in degree n is the motive $H^n(X,\mathbb{Q})$. All objects and morphisms in this complex come via the canonical lift $\widetilde{\rho}$ from the quiver $\mathrm{Q_c}(k)$, hence applying the functor $\mathrm{R_A}$ to it shows that $\mathrm{R_A}(H^n(X,\mathbb{Q}))$ is the homology in degree n of the complex

$$\cdots \longrightarrow H^*(\mathfrak{h}([\widetilde{X}_p, \widetilde{X}_{p-1}])) \longrightarrow H^*(\mathfrak{h}([\widetilde{X}_{p+1}, \widetilde{X}_p])) \longrightarrow \cdots \tag{34}$$

in $\mathbf{A}(k)$. Functoriality of the weight complex yields morphisms of complexes in $\mathbf{A}(k)$

$$\mathfrak{h}(X) \longleftarrow \mathfrak{h}(\widetilde{X}) \longrightarrow \mathfrak{h}(\widetilde{X}_p) \longleftarrow \mathfrak{h}([\widetilde{X}_p, \widetilde{X}_{p-1}])$$

which induce a natural morphism from $\mathfrak{h}^n(X) \subseteq \mathfrak{h}(X)$ to the homology in degree n of (34), which is a direct factor of $H^*(\mathfrak{h}([\widetilde{X}_n, \widetilde{X}_{n-1}]))$. This morphism $\mathfrak{h}^n(X) \longrightarrow \mathrm{R_A}(H^n(X,\mathbb{Q}))$ is an isomorphism, since it is an isomorphism after Betti realisation, that is, on singular cohomology. The additional statement (2) follows by applying Hodge realisations everywhere, and is as well true for ℓ-adic realisations. $\square$

Proof of statement (2) of Theorem 8.2. We start by constructing a functor $\mathbf{A}(k) \longrightarrow \mathbf{M}(k)$. Let X be a smooth projective variety of dimension n over k. I claim that any linear map

$$\mathbb{Q} \cong H^0(\mathrm{spec}(k)) \longrightarrow H^{2d}(X, \mathbb{Q})(d)$$

sending 1 to a motivated cycle is a morphism in $\mathbf{M}(k)$. To check this, notice first that there is a well defined cycle class map

$$\mathrm{CH}^n(X) \longrightarrow \mathrm{Hom}_{\mathbf{M}(k)}(\mathbb{1}, H^{2n}(X, \mathbb{Q})(n))$$

sending the class of a codimension n subvariety $Z \subseteq X$ to the morphism

$$\mathbb{1} \cong H^{2n-2d}(Z)(n-d) \longrightarrow H^{2n-2d}(X)(n-d) \cong H^{2d}(X)(d)$$

in $\mathbf{M}(k)$. The isomorphism on the left was explained in Proposition 5.4, and the isomorphism on the right is given by Poincaré duality. Consider now a motivated cycle $c \in H^{2d}(X, \mathbb{Q})(d)$, and write it as $c = \pi_*(a \cdot *_L(b))$ for some smooth projective variety Y and algebraic cycles a and b on $X \times Y$ as explained in 8.8. We have seen that a and b are images of 1 under morphisms of motives $H^0(\mathrm{spec}(k)) \longrightarrow H^{2i}(X \times Y, \mathbb{Q})(i)$. The Lefschetz isomorphism is an isomorphism in $\mathbf{M}(k)$ because cup products are morphisms in $\mathbf{M}(k)$ and a morphism in $\mathbf{M}(k)$ is an isomorphism if and only if the corresponding morphism of vector spaces is an isomorphism. It follows that also c is the image of 1 of a morphism in $\mathbf{M}(k)$ as claimed.

There is a canonical functor $\iota : \mathbf{A}(k) \longrightarrow \mathbf{M}(k)$ defined as follows: Given a smooth projective variety X of dimension n over k and a motivated projector $e \in H^{2n}(X \times X, \mathbb{Q})$, we set

$$\iota(e\mathfrak{h}(X)) = \ker\big(e : H^n(X, \mathbb{Q}) \longrightarrow H^n(X, \mathbb{Q})\big)$$

which makes sense because e is in the image of a morphism $\mathbb{Q}(-n) \longrightarrow H^{2n}(X \times X, \mathbb{Q})$, hence induces a morphism of motives $H^n(X, \mathbb{Q}) \longrightarrow H^n(X, \mathbb{Q})$. Morphisms in $\mathbf{A}(k)$ are sent to the morphisms in $\mathbf{M}(k)$ which are given by the very same linear maps, which are indeed morphisms of motives for the same reason. The so defined functor $\iota : \mathbf{A}(k) \longrightarrow \mathbf{M}(k)$ is strictly compatible with Betti realisation and in particular is faithful and exact. The following diagram

$$\begin{tikzcd} \mathbf{A}(k) \arrow[rr, "\iota"] \arrow[dr, "\mathrm{R_B}"'] & & \mathbf{M}(k) \arrow[dl, "\mathrm{R_B}"] \\ & \mathbf{Modf}_{\mathbb{Q}} & \end{tikzcd}$$

commutes. The functor ι is also strictly compatible with tensor products for the good, sign modified commutativity constraint on $\mathbf{A}(k)$. We also notice that the image of ι is contained in the full subcategory of semisimple objects of $\mathbf{M}(k)$, indeed, we already know that for every smooth projective variety X the motive $H^n(X, \mathbb{Q})$ is pure, hence semisimple by statement (1) of Theorem 8.2. I claim that the induced functor

$$\iota : \mathbf{A}(k) \longrightarrow \mathbf{M}_{\mathrm{ss}}(k)$$

is an equivalence of categories. An inverse to ι is given by the composite $\kappa : \mathbf{M}_{\mathrm{ss}}(k) \longrightarrow \mathbf{A}(k)$ of the inclusion $\mathbf{M}_{\mathrm{ss}}(k) \longrightarrow \mathbf{M}(k)$ and the functor $\mathrm{R_A}$ of Proposition 8.11. The functors ι and κ are both faithful. Since homomorphism sets in $\mathbf{A}(k)$ and $\mathbf{M}(k)$ are finite dimensional vector spaces, both, $\iota \circ \kappa$ and $\kappa \circ \iota$, are indeed fully faithful. For every smooth proper variety X over k, we have natural isomoprhisms $\kappa(\iota(\mathfrak{h}^n(X)(i))) \cong \mathfrak{h}^n(X)(i)$ in $\mathbf{A}(k)$ and $\iota(\kappa(H^n(X)(i))) \cong H^n(X)(i)$ in $\mathbf{M}(k)$ by construction of ι and statement (1) of Proposition 8.12. Since every object of $\mathbf{A}(k)$ is a direct factor of $\mathfrak{h}^n(X)(i)$ and every object of $\mathbf{M}_{\mathrm{ss}}(k)$ is a direct factor of $H^n(X)(i)$ for some smooth proper X, this shows that the functors $\iota \circ \kappa$ and $\kappa \circ \iota$ are both equivalent to the respective identitiy functors. $\square$

9. Motives with finite coefficients, finite motives and Artin motives

We call Artin motives those motives which which can be obtained by taking sums, quotients or subobjects of motives of the form $H^0([X,Y])$. The ℓ-adic realisation of an Artin motive is a Galois representation with finite image, and the Hodge realisation of an Artin motive is a Hodge structure with trivial Mumford-Tate group. Artin motives also have finite motivic Galois groups. Conjecturally, also the converses hold.

In this section we will show (Theorem 9.8) that motives with coefficients in a finite field are all Artin motives. More precisely, we show that for every finite field $\mathbb{F}$, the category $\mathbf{M}(k)_{\mathbb{F}}$ is equivalent via étale realisation to the category finite dimensional $\mathbb{F}$-linear continuous representations of the absolute Galois group $\mathrm{Gal}(\overline{k}|k)$, for the algebraic closure $\overline{k}$ of k in $\mathbb{C}$.

Next up, we study the relation between the category $\mathbf{M}(k)_{\mathbb{F}_\ell}$ and the full subcategory of $\mathbf{M}(k)_{\mathbb{Z}}$ of objects annihilated by ℓ. It turns out that these categories are equivalent. As a consequence, we find that the motivic fundamental group of an algebraically closed field is connected, as a group scheme over $\mathbb{Z}$ (Corollary 9.16). The killjoy is the fact that we are considering motives with integral coefficients and the corresponding group scheme over $\mathbb{Z}$, and not motives with rational coefficients and the corresponding group scheme over $\mathbb{Q}$: A flat group scheme over $\mathbb{Z}$ can be connected (it has no nontrivial finite flat quotient) without its generic fibre being connected.

- 9.1. We fix a subfield k of $\mathbb{C}$, denote by $\overline{k}$ its algebraic closure in $\mathbb{C}$ and set $\mathrm{Gal}_k := \mathrm{Gal}(\overline{k}|k)$. Moreover, we fix a Dedekind ring R. The category

$$\mathbf{Rep}(\mathrm{Gal}_k, R)$$

is the category of finitely generated discrete R-modules equipped with a continuous Gal_k-action, with equivariant linear maps as morphisms. Writing $R[\mathrm{Gal}_k]$ for the group proalgebra of the profinite group Gal_k, the category $\mathbf{Rep}(\mathrm{Gal}_k, R)$ is the same as the category of continuous $R[\mathrm{Gal}_k]$-modules which are finitely generated as R-modules. For every variety X over k, let us write $\pi_0(X)$ for the scheme of geometrically connected components of X. It is a finite étale scheme over k whose complex points are the connected components of the topological space $X(\mathbb{C})$. Lastly, we fix a prime number ℓ, and we write $\mathbb{F}$ or $\mathbb{F}_\ell$ for the finite field $\mathbb{Z}/\ell\mathbb{Z}$.

- 9.2. We denote by $\mathrm{Q}_0(k)$ the full subquiver of $\mathrm{Q}(k)$ whose objects are those of the form $[X,Y,0,0]$ for any k-variety X and closed suvariety Y, and by $\mathrm{Q}_{\text{ét}}(k)$ the full subquiver of $\mathrm{Q}(k)$ whose objects are those of the form $[X,Y,0,0]$ where X is finite and étale over k. Notice that $\mathrm{Q}_{\text{ét}}(k)$ is a subquiver of the quiver of cellular pairs $\mathrm{Q}_{\mathrm{c}}(k)$. We denote by $\mathbf{M}_0(k)_R$ and $\mathbf{M}_{\text{ét}}(k)_R$ the respective linear hulls of the restriction of the standard quiver representation $\rho : \mathrm{Q}(k) \longrightarrow \mathbf{Modf}_R$. We call *canonical* *functors* those induced by the quiver inclusions as follows.

$$\begin{tikzcd} \mathrm{Q}_{\text{ét}}(k) \arrow[r] \arrow[d] & \mathrm{Q}_0(k) \arrow[d] \\ \mathrm{Q}_{\mathrm{c}}(k) \arrow[r] & \mathrm{Q}(k) \end{tikzcd} \qquad \begin{tikzcd} \mathbf{M}_{\text{ét}}(k)_R \arrow[r] \arrow[d] & \mathbf{M}_0(k)_R \arrow[d] \\ \mathbf{M}_{\mathrm{c}}(k)_R \arrow[r] & \mathbf{M}(k)_R \end{tikzcd}$$

The canonical functor $\mathbf{M}_{\mathrm{c}}(k)_R \longrightarrow \mathbf{M}(k)_R$ is an equivalence of categories by Corollary 3.7.

Lemma 9.3. *The canonical functor $\mathbf{M}_{\text{ét}}(k)_R \longrightarrow \mathbf{M}_0(k)_R$ is an equivalence of categories. For every variety X over k and closed subvariety $Y \subseteq X$, let the Galois group Gal_k act on on the finitely generated free R-module*

$$\rho([X,Y,0,0]) = H^0([X,Y],R) = \ker(R^{\pi_0(X)(\mathbb{C})} \longrightarrow R^{\pi_0(Y)(\mathbb{C})})$$

via its action on $\pi_0(X)(\mathbb{C}) = \pi_0(X)(\overline{k})$. The functor induced by this action and the universal property 1.11 of linear hulls

$$\mathbf{M}_0(k)_R \longrightarrow \mathbf{Rep}(\mathrm{Gal}_k, R) \tag{35}$$

is an equivalence of categories.

Proof. To show that $\mathbf{M}_{\text{ét}}(k)_R \longrightarrow \mathbf{M}_0(k)_R$ is an equivalence of categories, we must according to Lemma 1.14 find a quiver representation $\mathrm{Q}_0(k) \longrightarrow \mathbf{M}_{\text{ét}}(k)$ such that

$$\begin{tikzcd} \mathrm{Q}_{\text{ét}}(k) \arrow[r] \arrow[d, "\subseteq"'] & \mathbf{M}_{\text{ét}}(k) \arrow[d, "\mathrm{can}"] \\ \mathrm{Q}_0(k) \arrow[r] \arrow[ur, "\lambda"] & \mathbf{M}_0(k) \end{tikzcd} \tag{36}$$

commutes up to natural isomorphisms. We define λ by

$$\lambda([X,Y,0,0]) = \ker\left(H^0([\pi_0(X),\varnothing]) \longrightarrow H^0([\pi_0(Y),\varnothing])\right)$$

on objects and in the evident way on morphisms of type (a) in Definition 2.2 which are given by morphisms between varieties. Note that $\mathrm{Q}_0(k)$ does not contain morphisms of type (b) and (c). The upper triangle in (36) trivially commutes. Also the lower triangle commutes, indeed, for every pair $[X,Y]$ the natural morphism $X \longrightarrow \pi_0(X)$ induces a morphism

$$\ker\left(H^0([\pi_0(X),\varnothing],R) \longrightarrow H^0([\pi_0(Y),\varnothing],R)\right) \longrightarrow H^0([X,Y],R)$$

in $\mathbf{M}_0(k)$ which is an isomorphism of R-modules, hence an isomorphism in $\mathbf{M}_0(k)$. This proves the first statement of the Lemma. For the the second statement, it suffices now to prove that the functor $\mathbf{M}_{\text{ét}}(k) \longrightarrow \mathbf{Rep}(\mathrm{Gal}_k, R)$ sending $H^0([X,\varnothing],R)$ to $R^{\pi_0(X)(\overline{k})}$ is an equivalence of categories. This is clear, since the quiver $Q_{\text{ét}}(k)$ is up to useless decoration the category of finite étale k-schemes, hence of finite continuous Gal_k-sets, and $\mathbf{Rep}(\mathrm{Gal}_k, R)$ is the linear hull of the Quiver representation

$$\{\text{Finite continuous } \mathrm{Gal}_k\text{-sets}\} \longrightarrow \mathbf{Rep}(\mathrm{Gal}_k, R)$$

sending an Gal_k-sets X to the free R-module R^X. $\square$

- 9.4. We now come to $\mathbb{F}$-linear categories of motives, where $\mathbb{F} = \mathbb{Z}/\ell\mathbb{Z}$ is the finite field with ℓ elements for a fixed prime number ℓ. The following two categories will be considered:

(1) The linear hull of the quiver representation $\rho_\ell : \mathrm{Q_c}(k) \longrightarrow \mathbf{Modf}_\mathbb{F}$ given by

$$\rho_\ell : [X, Y, n, i] \longmapsto H^n([X, Y], \mathbb{F})(i) \cong H^n([X, Y], \mathbb{Z})(i) \otimes \mathbb{F}$$

which is the category we denoted by $\mathbf{M}(k)_\mathbb{F}$ or by $\langle \mathrm{Q_c}(k), \rho \otimes \mathbb{F}\rangle$. The natural isomorphism stems from the fact that $[X, Y]$ is a cellular pair, so that in particular its cohomology is torsion free.

(2) The full subcategory of $\mathbf{M}(k)_\mathbb{Z}$, that is, of the linear hull of the quiver representation $\rho : \mathrm{Q_c}(k) \longrightarrow \mathbf{Modf}_\mathbb{Z}$ given by

$$\rho : [X, Y, n, i] \longmapsto H^n([X, Y], \mathbb{Z})(i)$$

consisting of those objects whose underlying $\mathbb{Z}$-module is annihilated by ℓ. This is the category $\langle \mathrm{Q}(K), \rho\rangle \otimes \mathbb{F} = \mathbf{M}(k)_\mathbb{Z} \otimes \mathbb{F}$.

There are as well the analogues of (1) and (2) for the general quiver of relative varieties $\mathrm{Q}(k)$, but one has to be careful in (1), since there is no longer such an isomorphism.

- 9.5. In 1.19 we have shown that there is a canonical functor $\mathbf{M}(k)_\mathbb{F} \longrightarrow \mathbf{M}(k)_\mathbb{Z} \otimes \mathbb{F}$ induced by a canonical morphism of proalgebras

$$\eta : \mathrm{End}(\rho) \otimes \mathbb{F} \longrightarrow \mathrm{End}(\rho_\ell)$$

which is not an isomorphism on the nose: Neither of the sufficient conditions of Proposition 1.20 is satisfied. However, since the quiver representation $\rho : \mathrm{Q_c}(k) \longrightarrow \mathbf{Modf}_\mathbb{Z}$ takes values in free $\mathbb{Z}$–modules, the conditions of Propostion 1.21 are satisfied, hence η is injective. Theorem 9.9 states that η is indeed an isomorphism. In fact, both of these algebras are isomorphic to the group algebra $\mathbb{F}[\mathrm{Gal}_k]$ via étale realisation.

- 9.6. Both categories presented in 9.4 come equipped with étale realisation functors. For the category $\mathbf{M}(k)_\mathbb{F}$ we have constructed it in Corollary 2.8. On the category $\mathbf{M}(k)_\mathbb{Z} \otimes \mathbb{F}$ we define the étale realisation functor as follows: In 2.8 we have defined the étale realisation functor

$$\mathrm{R}_\ell : \mathbf{M}(k)_{\mathbb{Z}_\ell} \longrightarrow \mathbf{Rep}(\mathrm{Gal}_k)_{\mathbb{Z}_\ell}$$

with values in the category of continuous ℓ–adic Galois representations for the ℓ–adic topology on $\mathbb{Z}_\ell$. Since $\mathbb{Z}_\ell$ is a flat $\mathbb{Z}$-algebra, we have canonical equivalences of categories

$$\mathbf{M}(k)_\mathbb{Z} \otimes \mathbb{F} \cong \mathbf{M}(k)_\mathbb{Z} \otimes \mathbb{Z}_\ell \otimes \mathbb{F} \cong \mathbf{M}(k)_{\mathbb{Z}_\ell} \otimes \mathbb{F}$$

where on the right hand side now stands the category of motives with $\mathbb{Z}_\ell$-coefficients whose underlying $\mathbb{Z}_\ell$-module is annihilated by ℓ. The étale realisation functor R_ℓ restricts to a functor

$$\mathrm{R}_\ell : \mathbf{M}(k)_\mathbb{Z} \otimes \mathbb{F} \longrightarrow \mathbf{Rep}(\mathrm{Gal}_k, \mathbb{F})$$

as we wanted. Notice that the canonical functor $\mathbf{M}(k)_\mathbb{F} \longrightarrow \mathbf{M}(k)_\mathbb{Z} \otimes \mathbb{F}$ commutes with étale realisation functors: Indeed, for every cellular pair $[X, Y]$ the canonical isomorphism of $\mathbb{F}$-modules $H^n([X, Y], \mathbb{F})(i) \cong H^n([X, Y], \mathbb{Z}_\ell)(i) \otimes \mathbb{F}$ is Galois equivariant, essentially by definition of étale ℓ-adic cohomology.

Lemma 9.7. *There is an isomorphism $\mathbb{1}(\ell-1) \cong \mathbb{1}$ in $\mathbf{M}(k)_{\mathbb{F}}$, that is to say, an isomorphism*

$$H^1(\mathbb{G}_m, \mathbb{F})^{\otimes(\ell-1)} \xrightarrow{\cong} H^0(\operatorname{spec} k, \mathbb{F})$$

and if k contains a primitive ℓ-th root of unity, there is even an isomorphism $\mathbb{1}(1) \cong \mathbb{1}$.

Proof. Let us write μ_ℓ for the closed subvariety of $\mathbb{G}_m$ consisting of the ℓ-th roots of unity. There is a morphism of relative varieties $[\mathbb{G}_m, \mu_\ell] \longrightarrow [\mathbb{G}_m, \{1\}]$ given by $z \longmapsto z^\ell$ on points. The induced morphism of long exact sequences reads

$$\begin{tikzcd}
H^0(\mathbb{G}_m,\mathbb{F}) \arrow[r, "\cong"] \arrow[d, equal] & H^0(\{1\},\mathbb{F}) \arrow[r, "0"] \arrow[d, hook] & H^1([\mathbb{G}_m,\{1\}],\mathbb{F}) \arrow[r, "\cong"] \arrow[d, hook] & H^1(\mathbb{G}_m,\mathbb{F}) \arrow[d, "0"] \\
H^0(\mathbb{G}_m,\mathbb{F}) \arrow[r, hook] & H^0(\mu_\ell,\mathbb{F}) \arrow[r] & H^1([\mathbb{G}_m,\mu_\ell],\mathbb{F}) \arrow[r, two heads] & H^1(\mathbb{G}_m,\mathbb{F})
\end{tikzcd}$$

with injective and surjective morphisms as indicated. The right hand vertical morphism is multiplication-by-ℓ, hence zero since we are in characteristic ℓ. This way, $H^1(\mathbb{G}_m, \mathbb{F})$ gets identified with a one dimensional subquotient of $H^0(\mu_\ell, \mathbb{F})$. Let $\zeta \in \mathbb{C}$ be a primitive ℓ-th root of unity. Elements of $H^0(\mu_\ell, \mathbb{F})$ are formal $\mathbb{F}$-linear combinations of $1, \zeta, \ldots, \zeta^{\ell-1}$, and the the circle segments from ζ^i to ζ^{i+1} form a basis of the homology group $H_1([\mathbb{G}_m, \mu_\ell], \mathbb{F})$. Let $\gamma_0, \ldots, \gamma_{\ell-1}$ be the dual basis. With respect to these bases, the diagram above is the following:

$$\begin{tikzcd}
\mathbb{F} \arrow[r, equal] \arrow[d, equal] & \mathbb{F} \arrow[r, "0"] \arrow[d, "\mathrm{diag}"'] & \mathbb{F} \arrow[r, "\cong"] \arrow[d, "\mathrm{diag}"'] & \mathbb{F} \arrow[d, "0"] \\
\mathbb{F} \arrow[r, "\mathrm{diag}"'] & \mathbb{F}^\ell \arrow[r, "\partial"'] & \mathbb{F}^\ell \arrow[r, "\mathrm{sum}"'] & \mathbb{F}
\end{tikzcd} \qquad \partial\Big(\sum_{i\in\mathbb{Z}/\ell\mathbb{Z}} a_i[\zeta^i]\Big) = \sum_{i\in\mathbb{Z}/\ell\mathbb{Z}} a_i([\gamma_i] - [\gamma_{i+1}])$$

and the one dimensional subspace of $H^0(\mu_\ell, \mathbb{F})/H^0(\mathbb{G}_m, \mathbb{F})$ generated by the class of the element

$$m = \sum_{i=0}^{\ell-1} i[\zeta^i] \quad \in H^0(\mu_\ell, \mathbb{F})$$

corresponds to the motive $H^1(\mathbb{G}_m, \mathbb{F})$. Now we look at $H^0(\mu_\ell, \mathbb{F})$ as an object of the category $\mathbf{M}_0(k)_{\mathbb{F}}$. The Galois group acts on $H^0(\mu_\ell, \mathbb{F})$ via the cyclotomic character $\chi : \mathrm{Gal}_k \longrightarrow (\mathbb{Z}/\ell\mathbb{Z})^*$ according to

$$g^* \sum_{i\in\mathbb{Z}/\ell\mathbb{Z}} a_i[\zeta^i] = \sum_{i\in\mathbb{Z}/\ell\mathbb{Z}} a_i[\zeta^{i\chi(g)}] = \sum_{i\in\mathbb{Z}/\ell\mathbb{Z}} a_{i\chi(g)^{-1}}[\zeta^i]$$

and we have in particular $g^*m = \chi(g)^{-1}m$. Hence $m\mathbb{F}$ is an eigenspace for the Galois action, and by Lemma 9.3 this implies that $m\mathbb{F}$ is a subobject of $H^0(\mu_\ell, \mathbb{F})$ in the category $\mathbf{M}_0(k)_{\mathbb{F}}$. The vector $m^{\otimes(\ell-1)} \in H^0(\mu_\ell, \mathbb{F})^{\otimes(\ell-1)}$ is fixed under Gal_k, hence $(m\mathbb{F})^{\otimes(\ell-1)}$ is isomorphic to $H^0(\operatorname{spec} k, \mathbb{F})$ in $\mathbf{M}_0(k)_{\mathbb{F}}$, again by Lemma 9.3. The image in $\mathbf{M}(k)_{\mathbb{F}}$ of any such isomorphism is what we sought. If k contains a primitive ℓ-th root of unity, already m is fixed. □

Theorem 9.8. *The canonical functor $\mathbf{M}_0(k)_{\mathbb{F}} \longrightarrow \mathbf{M}(k)_{\mathbb{F}}$ and the étale realisation functor $\mathbf{M}(k)_{\mathbb{F}} \longrightarrow \mathbf{Rep}(\mathrm{Gal}_k, \mathbb{F})$ are equivalences of categories.*

Proof. By Lemma 9.3, it suffices to show that the canonical functor $\mathbf{M}_0(k)_\mathbb{F} \longrightarrow \mathbf{M}(k)_\mathbb{F}$ is an equivalence of categories. According to Lemma 1.14, a quiver representation $\lambda : \mathrm{Q}(k) \longrightarrow \mathbf{M}_0(k)_\mathbb{F}$ needs to be constructed, such that the diagram

$$\begin{tikzcd} \mathrm{Q}_0(k) \arrow[r] \arrow[d, "\subseteq"'] & \mathbf{M}_0(k)_\mathbb{F} \arrow[d, "\mathrm{can}"] \\ \mathrm{Q}(k) \arrow[r] \arrow[ur, "\lambda"] & \mathbf{M}(k)_\mathbb{F} \end{tikzcd} \tag{37}$$

commutes up to natural isomorphisms. Fix an object $[X,Y,n,i]$ of $\mathrm{Q}(k)$. For every finite étale covering $\mathfrak{U} = (U_i)_{i\in I}$ of X define a complex in $\mathbf{M}_0(k)_\mathbb{F}$ by

$$\check{C}_\mathfrak{U}([X,Y]) = \Big[\prod_{i\in I} H^0([U_i,V_i],\mathbb{F}) \longrightarrow \prod_{i,j\in I} H^0([U_{ij},V_{ij}],\mathbb{F}) \longrightarrow \cdots\Big] \tag{38}$$

with $U_{ij} := U_i \times_X U_j$ and $V_i = Y \times_X U_i$ and so forth. We define λ by

$$\lambda([X,Y,n,i]) = \operatorname*{colim}_{\mathfrak{U}} H^n(\check{C}_\mathfrak{U}([X,Y]))(i)$$

on objects, the colimit running over all finite étale coverings of X, and in the straightforward way on morphisms. Commutativity of the upper triangle in (37) is clear. It remains to check that the lower triangle commutes as well. We can think of $[U_\bullet, V_\bullet] \longrightarrow [X,Y]$ as an augmented simplicial relative variety, hence there is a canonical morphism of motives

$$H^n(\check{C}_\mathfrak{U}([X,Y])) \longrightarrow H^n([X,Y])$$

by Theorem 4.4, and hence a morphism of motives $\lambda([X,Y,n,i]) \longrightarrow H^n([X,Y],\mathbb{F})(i)$. It remains to show that this morphism is an isomorphism of $\mathbb{F}$-modules. Write $\mathcal{F}$ for the sheaf $\beta_!\beta^*\underline{\mathbb{F}}_X$ on $X(\mathbb{C})$, where β is the inclusion of the complement of $Y(\mathbb{C})$ into $X(\mathbb{C})$. Since $\mathcal{F}$ is a torsion sheaf as a sheaf of groups, there is a natural isomorphism

$$H^n((X_{\overline{k}})_{\text{ét}}, \mathcal{F}) \longrightarrow H^n(X(\mathbb{C}),\mathcal{F})$$

according the comparison theorem of Artin and Grothendieck. Etale cohomology with torsion coefficients can be computed using Čech-complexes with respect to finite étale covers of X, hence so can $H^n(X(\mathbb{C}),\mathcal{F})$. The Čhech-complex corresponding to the covering $\mathfrak{U} = (U_i)_{i\in I}$ of X in terms of sheaf cohomology reads

$$\check{C}_\mathfrak{U}(X,\mathcal{F}) = \Big[\prod_{i\in I} H^0(U_i,\mathcal{F}) \longrightarrow \prod_{i,j\in I} H^0(U_{ij},\mathcal{F}) \longrightarrow \cdots\Big]$$

and is indeed the same as (38) by definition of relative cohomology. □

Theorem 9.9. *The canonical functor $\mathbf{M}(k)_\mathbb{F} \longrightarrow \mathbf{M}(k)_\mathbb{Z} \otimes \mathbb{F}$ is an equivalence of categories.*

Proof. The composition of the canonical functor $\mathbf{M}(k)_\mathbb{F} \longrightarrow \mathbf{M}(k)_\mathbb{Z} \otimes \mathbb{F}$ with the étale realisation functor $\mathbf{M}(k)_\mathbb{Z} \otimes \mathbb{F} \longrightarrow \mathbf{Rep}(\mathrm{Gal}_k, \mathbb{F})$ is the étale realisation functor on $\mathbf{M}(k)_\mathbb{F}$, hence is an equivalence of categories by Theorem 9.8.

$$\begin{tikzcd} \mathbf{M}_0(k)_\mathbb{F} \arrow[r, "\mathrm{can}"] \arrow[rr, bend right, "\cong"'] & \mathbf{M}(k)_\mathbb{Z} \otimes \mathbb{F} \arrow[r, "\mathrm{R}_\ell"] & \mathbf{Rep}(\mathrm{Gal}_k, \mathbb{F}) \end{tikzcd}$$

These categories are categories of modules for proalgebras, and the functors between them are induced by morphisms between these proalgebras as follows.

$$\begin{tikzcd} \mathrm{End}(\rho_\ell) & \mathrm{End}(\rho)\otimes\mathbb{F} \arrow[l, "\eta"'] & \mathbb{F}[\mathrm{Gal}_k] \arrow[l] \arrow[ll, bend left, "\cong"] \end{tikzcd}$$

From Proposition 1.21 we know that η is injective, hence all morphisms must be isomorphisms. $\square$

We now come back to coefficients of characteristic zero. We have seen in Lemma 9.3 that the subcategory $\mathbf{M}_0(k)_\mathbb{Q}$ of $\mathbf{M}(k)_\mathbb{Q}$ is equivalent to the category of $\mathbb{Q}$-linear continuous Galois representations. These are the *Artin motives*. The problem we face now is the following: Being a neutral $\mathbb{Q}$-linear tannakian category, $\mathbf{M}(k)_\mathbb{Q}$ is equivalent to the the category of representations of a group scheme $G_{\mathrm{mot}}(k)$ over $\mathbb{Q}$. If M is an Artin motive, then the corresponding representation $G_{\mathrm{mot}}(k) \longrightarrow \mathrm{GL}_{\mathrm{R_B}(M)}$ factors over a finite quotient of $G_{\mathrm{mot}}(k)$. One might guess that every representation of $G_{\mathrm{mot}}(k)$ with finite image corresponds to an Artin motive, but this is by no means clear at the moment.

If we work with a coefficient ring R which is not field, it makes no longer sense to speak about representations, as explained in 7.5. The right substitute is the notion of *Weil-finite* objects, which makes sense in any R-linear abelian monoidal category. If R is a field, then Weil-finite motives correspond indeed to those representations of the Tannakian fundamental group of $\mathbf{M}(k)_R$ which factor over a finite group. The definition of Weil-finiteness is then also equivalent to the one introduced by Esnault and Hai in [EH08] from where I borrowed the terminology.

Definition 9.10. We say that a motive $M \in \mathbf{M}(k)_R$ is *Weil-finite* if the full monoidal closed abelian subcategory of $\mathbf{M}(k)_R$ generated by M has a projective generator. We denote by $\mathbf{M}_{\mathrm{Wf}}(k)_R$ the full subcategory of $\mathbf{M}(k)_R$ whose objects are the Weil-finite motives.

- 9.11. By definition, $\mathbf{M}(k)_R$ is the category of continuous $\mathrm{End}(\rho)$-modules which are finitely generated as R-modules. A full abelian R-linear subcategory $\mathbf{C}$ of $\mathbf{M}(k)_R$ is, from this point of view, the category of continuous E-modules which are finitely generated as R-modules for a quotient E of $\mathrm{End}(\rho)$. The algebra E can be seen either as the endomorphism ring of the Betti realisation functor $\mathrm{R_B} : \mathbf{C} \longrightarrow \mathbf{Modf}_R$, or as the quotient of $\mathrm{End}(\rho)$ by the common annihilator of all objects of $\mathbf{C}$. The subcategory $\mathbf{C}$ has a projective generator if and only if E is finitely generated as an R-module. If $\mathbf{C}$ is a monoidal closed subcategory, then E is a Hopf algebra quotient of $\mathrm{End}(\rho)$.

- 9.12. The realisations of a Weil-finite motive M in various tannakian realisation categories are then objects with the same property. The canonical functor $\mathbf{M}_0(k)_R \longrightarrow \mathbf{M}(k)_R$ factors over the subcategory of Weil-finite motives, Theorem 9.15 states that Artin motives and Weil-finite motives are indeed the same if $R = \mathbb{Z}$.

Lemma 9.13. *Let M be an object of $\mathbf{M}(k)_\mathbb{Z}$ whose underlying $\mathbb{Z}$-module is annihilated by ℓ. Then M is Weil-finite.*

Proof. The tensor category of Galois representations generated by M has a projective generator, hence so has the tensor category of motives generated by M. $\square$

Lemma 9.14. *Let $f : E \longrightarrow F$ be a morphism of flat $\mathbb{Z}_\ell$-algebras which are finitely generated as $\mathbb{Z}_\ell$-modules. If the change-of-structure functor*

$$f^* : \{\textit{Finite } F\textit{-modules}\} \longrightarrow \{\textit{Finite } E\textit{-modules}\}$$

is an equivalence of categories, then f is an isomorphism.

Proof. Recall the following fact: A morphism of finite R-algebras $f : E \longrightarrow F$ is an isomorphism if and only if the functor

$$f^* : \{\text{Finitely generated } F\text{-modules}\} \longrightarrow \{\text{Finitely generated } E\text{-modules}\}$$

is an equivalence of categories. To give a finite E-module which is annihilated by ℓ is the same as to give an $E \otimes \mathbb{Z}/\ell\mathbb{Z}$-module. We find an equivalence of categories

$$\{\text{Finitely generated } F \otimes \mathbb{Z}/\ell\mathbb{Z}\text{-modules}\} \longrightarrow \{\text{Finitely generated } E \otimes \mathbb{Z}/\ell\mathbb{Z}\text{-modules}\}$$

which we can either see as the restriction of f^* to these categories, or as change-of-structure for the algebra morphism $f \otimes \mathrm{id} : E \otimes \mathbb{Z}/\ell\mathbb{Z} \longrightarrow F \otimes \mathbb{Z}/\ell\mathbb{Z}$. As was pointed out at the beginning of the proof, this implies that $E \otimes \mathbb{Z}/\ell\mathbb{Z} \longrightarrow F \otimes \mathbb{Z}/\ell\mathbb{Z}$ is an isomorphism. By Nakayama's Lemma this can only be if f was an isomorphism. $\square$

Theorem 9.15. *The canonical functor $\mathbf{M}_0(k)_\mathbb{Z} \longrightarrow \mathbf{M}_{\mathrm{Wf}}(k)_\mathbb{Z}$ is an equivalence of categories.*

Proof. Let M be an object of $\mathbf{M}_{\mathrm{Wf}}(k)_\mathbb{Z}$, that is, a Weil-finite motive with integral coefficients. Let $\langle M\rangle^\otimes$ be the full abelian monoidal subcategory of $\mathbf{M}(k)_\mathbb{Z}$ generated by M. By definition of Weil-finiteness, the category $\langle M\rangle^\otimes$ is the category of E-modules for a cocommutative Hopf algebra E which is finitely generated as a $\mathbb{Z}$-module. This algebra E is a quotient of the Hopf proalgebra $\mathrm{End}(\rho)$. The whole category of Weil finite motives $\mathbf{M}_{\mathrm{Wf}}(k)_\mathbb{Z}$ is therefore the category of E_{Wf}-modules, where the pro Hopf-algebra

$$E_{\mathrm{Wf}} = \lim_i E_i$$

is a limit of cocommutative Hopf algebras E_i which are all finitely generated as a $\mathbb{Z}$-modules. We can assume that the transition maps $E_i \longrightarrow E_j$ are surjective.

Pick a prime number ℓ. The étale ℓ-adic realisation functor yields a functor

$$\mathrm{R}_\ell : \mathbf{M}_{\mathrm{Wf}}(k)_{\mathbb{Z}_\ell} \longrightarrow \mathbf{Rep}(\mathrm{Gal}_k, \mathbb{Z}_\ell)$$

corresponding to a morphism of proalgebras $f : \mathbb{Z}_\ell[\mathrm{Gal}_k] \longrightarrow E \otimes \mathbb{Z}_\ell$. By Lemma 9.13, finite motives annihilated by ℓ are Weil-finite. This morphism induces thus an equivalence of categories

$$\mathbf{M}_{\mathrm{Wf}}(k)_{\mathbb{Z}_\ell} \otimes \mathbb{F} = \mathbf{M}(k)_{\mathbb{Z}_\ell} \otimes \mathbb{F} \longrightarrow \mathbf{Rep}(\mathrm{Gal}_k, \mathbb{F})$$

by Theorem 9.9, hence f is an isomorphism by Lemma 9.14. The following composite of functors is an equivalence of categories by Lemma 9.3:

$$\begin{tikzcd} \mathbf{M}_0(k)_{\mathbb{Z}_\ell} \arrow[r, "\mathrm{can}"] \arrow[rr, bend right, "\cong"'] & \mathbf{M}_{\mathrm{Wf}}(k)_{\mathbb{Z}_\ell} \arrow[r, "\mathrm{R}_\ell"] & \mathbf{Rep}(\mathrm{Gal}_k, \mathbb{Z}_\ell) \end{tikzcd}$$

We have seen that the étale realisation R_ℓ is also an equivalence, hence so is the canonical functor. But the canonical functor in this diagram is obtained by tensoring with $\mathbb{Z}_\ell$ the change of structure functor of a morphism of $\mathbb{Z}$-proalgebras $E_{\mathrm{Wf}} \longrightarrow \mathbb{Z}[\mathrm{Gal}_k]$. By flatness, this morphism is itself an isomorphism, and hence yields the desired equivalence. $\square$

Corollary 9.16. *The canonical morphism $G_{\mathrm{mot}}(k)_{\mathbb{Z}} \longrightarrow \mathrm{Gal}_k$ is surjective, and its kernel is connected (admits no nontrivial finite flat quotient). In particular, if k is an algebraically closed field, then its motivic fundamental group $G_{\mathrm{mot}}(k)_{\mathbb{Z}}$ is connected.*

Proof. Set $G := G_{\mathrm{mot}}(k)_{\mathbb{Z}}$ and denote by G° the connected component of the unity of G. Since Gal_k, seen as a constant group scheme over $\mathbb{Z}$, is totally disconnected, the canonical morphism in the statement of the corollary induces a morphism of profinite group schemes

$$G/G^\circ \longrightarrow \mathrm{Gal}_k$$

of which we want to prove that it is an isomorphism. The morphism of algebras of regular functions in the opposite direction is dual to the morphism of proalgebras $E_{\mathrm{Wf}} \longrightarrow \mathbb{Z}[\mathrm{Gal}_k]$, which is an isomorphism by Theorem 9.15. $\square$

- 9.17. Suppose $k \subseteq \mathbb{C}$ is algebraically closed. Let $\mathbf{M}_{\mathbb{Q}}(k)^{\mathbb{Z}}$ denote the category whose objects are pairs (M, L) consisting of an object M of $\mathbf{M}(k)_{\mathbb{Q}}$, a finitely generated group L and an isomorphism of vector spaces $\mathrm{R}_{\mathrm{B}}(M) \cong L \otimes \mathbb{Q}$. We have canonical functors

$$\mathbf{M}(k)_{\mathbb{Z}} \xrightarrow{(*)} \mathbf{M}_{\mathbb{Q}}(k)^{\mathbb{Z}} \longrightarrow \mathbf{M}_{\mathbb{Q}}(k)$$

the left hand one sending a motive M with integral coefficients to the pair $(M_{\mathbb{Q}}, \mathrm{R}_{\mathrm{B}}(M))$, and the right hand one sending (M, L) to M. The functor $(*)$ is an equivalence of categories.

10. Galois descent

Let $k \subseteq k'$ be a Galois extension of subfields of $\mathbb{C}$. We establish the following short exact sequence of group schemes groups over $\mathbb{Q}$:

$$1 \longrightarrow G_{\mathrm{mot}}(k')_{\mathbb{Q}} \longrightarrow G_{\mathrm{mot}}(k)_{\mathbb{Q}} \longrightarrow \mathrm{Gal}(k'|k) \longrightarrow 1 \tag{39}$$

We concentrate throughout most of the section on the essential case where k' is a finite Galois extension. If k' is the algebraic closure of k in $\mathbb{C}$ we obtain the exact sequence of Theorem 1 in the introduction. The morphisms in the sequence (39) are constructed in 10.3 and 10.5.

- 10.1. We fix the following material for the section: A subfield k of $\mathbb{C}$, and a finite Galois extension $k' \subseteq \mathbb{C}$ of k with Galois group $G := \mathrm{Gal}(k'|k)$. All motives are tacitly understood to be motives with rational coefficients.

- 10.2. Given a variety X' over k', we write j_*X' for its Weil restriction to k. This means, j_*X' is the variety over k given by the same scheme as X', and whose structural morphism is obtained by composition with the morphism $j : \mathrm{spec}(k') \longrightarrow \mathrm{spec}(k)$ corresponding to the inclusion $k \longrightarrow k'$. Given a variety X over k, we write $j^*X := X_{k'} = X \times_{\mathrm{spec}\, k} \mathrm{spec}\, k'$ for the variety over k' obtained by base change along j. There are natural adjunction morphisms as follows $j^*j_*X' \longrightarrow X'$ and $X \longrightarrow j_*j^*X$. We extend the functors j^* and j_* in the evident way to morphisms of quivers

$$j_* : \mathrm{Q}(k') \longrightarrow \mathrm{Q}(k) \qquad \text{and} \qquad j^* : \mathrm{Q}(k) \longrightarrow \mathrm{Q}(k')$$

and notice that for $q \in \mathrm{Q}(k)$ and $q' \in \mathrm{Q}(k')$ the adjunction morphisms

$$q' \longrightarrow j^*j_*q' \qquad \text{and} \qquad j_*j^*q \longrightarrow q$$

are morphisms of type (a) in these quivers. Their direction has been reversed.

- 10.3. Let me start with defining the injective morphism appearing in (39). We consider the morphism of quiver representations

$$\begin{tikzcd} \mathrm{Q}(k) \arrow[rr, "j^*"] \arrow[dr, "\rho"'] & & \mathrm{Q}(k') \arrow[dl, "\rho'"] \\ & \mathbf{Modf}_{\mathbb{Q}} & \end{tikzcd} \qquad \rho(q) = \rho'(j^*q)$$

where ρ' is the standard quiver representation on the quiver $\mathrm{Q}(k')$. The diagram commutes, because of the equality $X(\mathbb{C}) = (j^*X)(\mathbb{C})$ that holds for every variety X over k. From this morphism of quiver representations we obtain a morphism of proalgebras as follows.

$$E' := \mathrm{End}(\rho' : Q(k') \longrightarrow \mathbf{Modf}_{\mathbb{Q}}) \quad \xrightarrow{\;j\;} \quad E := \mathrm{End}(\rho : Q(k) \longrightarrow \mathbf{Modf}_{\mathbb{Q}})$$

The morphism j is compatible with comultiplications on E and E', the reason is that the given quiver morphism $Q(k) \longrightarrow Q(k')$ sends cellular objects to cellular objects and is strictly compatible with the product morphism (27) which was used to construct tensor products of motives. Therefore, the continuous dual of j is a morphism of commutative Hopf algebras in the opposite direction, and yields a morphism of group schemes $G_{\mathrm{mot}}(k')_{\mathbb{Q}} \longrightarrow G_{\mathrm{mot}}(k)_{\mathbb{Q}}$ as we sought to construct.

Proposition 10.4. *The morphism $j : E' \longrightarrow E$ is injective.*

Proof. Let $e' \in E'$ be an endomorphism of ρ' with $j(e') = 0$. The endomorphism e' is given as a collection of compatible endomorphisms $(e'_p)_{p \in \mathrm{Q}(k')}$

$$e'_p : H^n([X(\mathbb{C}), Y(\mathbb{C})], \mathbb{Q})(i) \longrightarrow H^n([X(\mathbb{C}), Y(\mathbb{C})], \mathbb{Q})(i) \qquad p = [X, Y, n, i] \in \mathrm{Q}(k')$$

and to say that $j(e')$ is zero is to say that for each $q \in \mathrm{Q}(k)$ we have $e'_{j^*q} = 0$. For every $p = [X, Y, n, i] \in \mathrm{Q}(k')$ we may consider the morphism $p \longrightarrow j^*j_*p$ in $Q(k')$. On the one hand the induced morphism $\rho'(p) \longrightarrow \rho'(j^*j_*p)$ is surjective, indeed, for every variety X over k' the continuous map $X(\mathbb{C}) \longrightarrow (j^*j_*X)(\mathbb{C}) = X(\mathbb{C}) \times \mathrm{Hom}_k(k', \mathbb{C})$ is just the inclusion of a connected component. On the other hand we have $e'_{j^*j_*p} = 0$ by hypothesis, hence $e'_p = 0$. Since p was arbitrary, we find $e' = 0$. $\square$

- 10.5. The morphism $G_{\mathrm{mot}}(k)_{\mathbb{Q}} \longrightarrow G$ is induced by the algebra morphism $\chi : E \longrightarrow \mathbb{Q}[G]$, which in turn is obtained from restricting ρ to the subquiver of $Q(k)$ consisting of the single object

$$q_0 := [\mathrm{spec}(k'), \varnothing, 0, 0]$$

and all its endomorphisms. Here is an explicit description of how this works. Since k' is a subfield of $\mathbb{C}$, the k-linear complex embeddings of k' are canonically in bijection with elements of the group G. Thus, the elements of G form a canonical basis $\{\delta_g \mid g \in G\}$ of the vector space

$$\rho(q_0) = H^0([\mathrm{spec}(k'), \varnothing], \mathbb{Q})) = \mathrm{Maps}(\mathrm{Hom}_k(k', \mathbb{C}), \mathbb{Q})$$

with $\delta_g \in \rho(q_0)$ corresponding to the map sending an embedding $k' \longrightarrow \mathbb{C}$ to $1 \in \mathbb{Q}$ if it is the composition of $g : k' \longrightarrow k'$ followed by the inclusion $k' \longrightarrow \mathbb{C}$, and to zero otherwise. Elements of E are endomorphisms of the quiver representation $\rho : \mathrm{Q}(k) \longrightarrow \mathbf{Modf}_{\mathbb{Q}}$, so there is a canonical algebra morphism $E \longrightarrow \mathrm{End}_{\mathbb{Q}}(\rho(q_0))$, sending $(e_q)_{q \in \mathrm{Q}(k)}$ to e_{q_0}. The endomorphisms e_q must commute with morphisms in $\mathrm{Q}(k)$, and in particular e_{q_0} must commute with elements $h \in G$, seen as morphisms $h : q_0 \longrightarrow q_0$ in $\mathrm{Q}(k)$.

$$\rho(h) : \rho(q_0) \longrightarrow \rho(q_0) \qquad\qquad \sum_{g \in G} a_g \delta_g \longmapsto \sum_{g \in G} a_{g \circ h} \delta_g = \sum_{g \in G} a_g \delta_{g \circ h^{-1}}$$

It follows that $e_{q_0} : \rho(q_0) \longrightarrow \rho(q_0)$ must be of the form

$$e_{q_0} : \sum_{g \in G} a_g \delta_g \longmapsto \sum_{h \in G} \sum_{g \in G} \chi(e)_h a_g \delta_{hg}$$

for some element $\chi(e) = \sum_{h \in G} \chi(e)_h [h] \in \mathbb{Q}[G]$. Routine checking shows that $\chi : E \longrightarrow \mathbb{Q}[G]$ is a well defined algebra morphism, which is compatible with comultiplication and counit for the standard cocommutative comultiplication on $\mathbb{Q}[G]$.

Proposition 10.6. *The morphism $\chi : E \longrightarrow \mathbb{Q}[G]$ is surjective.*

Proof. We have seen this already in Corollary 9.16 in the case where G is the absolute Galois group of k. The case of a finite Galois group is an immediate consequence. □

Theorem 10.7. *The sequence of group schemes*

$$1 \longrightarrow G_{\mathrm{mot}}(k')_{\mathbb{Q}} \longrightarrow G_{\mathrm{mot}}(k)_{\mathbb{Q}} \longrightarrow \mathrm{Gal}(k'|k) \longrightarrow 1$$

where is exact.

Proof. We start by reformulating the exactness of the sequence in question in terms of morphisms of Hopf algebras. We have introduced morphisms

$$E' \xrightarrow{j} E \xrightarrow{\chi} \mathbb{Q}[G]$$

and injectivity of j and surjectivity of χ translate to injectifity of $G_{\mathrm{mot}}(k')_{\mathbb{Q}}$ and surjectivity of $G_{\mathrm{mot}}(k)_{\mathbb{Q}} \longrightarrow \mathrm{Gal}(k'|k)$. Exactness in the middle is more delicate. The morphism of group schemes $\chi : G_{\mathrm{mot}}(k)_{\mathbb{Q}} \longrightarrow G$ permits us to decompose $G_{\mathrm{mot}}(k)_{\mathbb{Q}}$ into a finite disjoint union of cosets, each of which is open and closed in $G_{\mathrm{mot}}(k)_{\mathbb{Q}}$. From the point of view that E is the algebra of distributions on the group $G_{\mathrm{mot}}(k)_{\mathbb{Q}}$ we see that each element $e \in E$ can be written in a unique way as

$$e = \sum_{g \in G} r(e, g) \tag{40}$$

where $r(e, g) \in E$ is a distribution supported on the coset $\chi^{-1}(g)$. In terms of the comultiplication on E, we obtain this decomposition as follows: For $e \in E$, we can write $(\mathrm{id}_E \otimes \chi)(c(e)) \in E \otimes \mathbb{Q}[G]$ as

$$(\mathrm{id}_E \otimes \chi)(c(e)) = \sum_{g \in G} r(e, g) \otimes [g] \tag{41}$$

for certain uniquely determined $r(e, g) \in E$. The relation (40) follows then from the fact that χ is compatible with counits, and the counit on $\mathbb{Q}[G]$ is the sum of coefficients. More explicitly, the equality $e = r(e, g)$ holds for $e = (e_q)_{q \in \mathrm{Q}(k)}$ if and only if the diagram

$$\begin{tikzcd}
H^n([X \times_k k', Y \times_k k'], \mathbb{Q})(i) = \rho(j_* j^* q) \arrow[r, "e_{j_* j^* q}"] \arrow[d, "\cong"'] & \rho(j_* j^* q) \arrow[d, "\cong"] \\
H^n([X, Y], \mathbb{Q})(i) \otimes H^0([\mathrm{spec}(k'), \varnothing], \mathbb{Q})(0) = \rho(q) \otimes \rho(q_0) \arrow[r, "e_q \otimes [g]"] & \rho(q) \otimes \rho(q_0)
\end{tikzcd}$$

commutes for every object $q = [X, Y, n, i]$ of $\mathrm{Q}(k)$. Exactness in the middle of the sequence in Theorem 10.7 translates to the following:

Claim. *The image of the morphism $j : E' \longrightarrow E$ is equal to the set $E(1_G)$ consisting of those elements $e \in E$ satisfying $e = r(e, 1_G)$.*

That the image of j is contained in $E(1_G)$ is clear. In order to show that $j : E' \longrightarrow E(1_G)$ is surjective, pick an element $e \in E(1_G)$, and let us construct an element $e' \in E'$ with $j(e') = e$ as follows.

For every $q \in \mathrm{Q}(k')$ we are given a morphism $e_{j_* q} : \rho(j_* q) \longrightarrow \rho(j_* q)$, which we can restrict to a morphism $f_q : \sigma(q) \longrightarrow \sigma(q)$ by Lemma **??**. It needs to be checked that the family $f = (f_q)_{q \in \mathrm{Q}(k')}$ is indeed an element of F, and that $j(f) = e$ holds. Both verifications are straightforward computations. □

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