# Exponential motives

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Abstract. Following ideas of Katz, Kontsevich, and Nori, we construct a neutral $\mathbb{Q}$-linear tannakian category of exponential motives over a subfield $k$ of the complex numbers by applying Nori's formalism to rapid decay cohomology, which one thinks of as the Betti realisation. We then introduce the de Rham realisation, as well as a comparison isomorphism between them. When $k$ is algebraic, this yields a class of complex numbers, *exponential periods*, including special values of the gamma and the Bessel functions, the Euler–Mascheroni constant, and other interesting numbers which are not expected to be periods of usual motives. In particular, we attach to exponential motives a Galois group which conjecturally governs all algebraic relations among their periods.

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CHAPTER 1

# Introduction

What motives are to algebraic varieties, *exponential motives* are to varieties endowed with a *potential*, that is, to pairs $(X, f)$ consisting of an algebraic variety $X$ over some field $k$ and a regular function $f$ on $X$. These objects have attracted considerable attention in recent years, especially in connection with mirror symmetry, where one seeks to associate with a Fano variety $Y$ a Landau-Ginzburg mirror $(X, f)$ in such a way that certain invariants of $Y$, *e.g.* the Hodge numbers, are reflected by the geometry of $f$, namely its critical locus. Our motivation is somewhat different: exponential motives provide a framework to deal with many interesting numbers which are not expected to be periods in the usual sense of algebraic geometry. Following ideas of Katz, Kontsevich, and Nori, we shall construct a $\mathbb{Q}$-linear neutral tannakian category of exponential motives over a subfield $k$ of the complex numbers, and compute a few examples of Galois groups. Classical results and conjectures of transcendence theory may then be interpreted—in the spirit of Grothendieck's period conjecture—as saying that the transcendence degree of the field generated by the periods of an exponential motive agrees with the dimension of its Galois group.

## 1.1. Exponential periods

1.1.1 (Two cohomology theories). — To get in tune, let us introduce two cohomology theories for varieties with a potential. The first one, rapid decay cohomology, appears implicitly in the classical study of the asymptotics of differential equations with irregular singularities. To our knowledge, it was first considered in a 1976 letter from Deligne to Malgrange [**DMR07**, p.17].

Given a real number $r$, let $S_r \subseteq \mathbb{C}$ denote the closed half-plane $\{\mathrm{Re}(z) \geqslant r\}$. If $X$ is a complex algebraic variety and $f\colon X \to \mathbb{C}$ a regular function, the *rapid decay homology* in degree $n$ of the pair $(X, f)$ is defined as the limit

$$H_n^{\mathrm{rd}}(X, f) = \lim_{r \to +\infty} H_n(X(\mathbb{C}), f^{-1}(S_r); \mathbb{Q}). \tag{1.1.1.1}$$

On the right-hand side stands the singular homology with rational coefficients of the topological space $X(\mathbb{C})$ relative to the closed subspace $f^{-1}(S_r)$, and the limit is taken with respect to the transition maps induced by the inclusions $f^{-1}(S_{r'}) \subseteq f^{-1}(S_r)$ for all $r' \geqslant r$. For big enough $r$, these maps are isomorphisms and the fibre $f^{-1}(r)$ is homotopically equivalent to $f^{-1}(S_r)$, so one may as well think of rapid decay homology as the homology of $X(\mathbb{C})$ relative to a general fibre. The reason for the name will become apparent soon.

With this definition settled, the $n$-th *rapid decay cohomology* group $H^n_{\mathrm{rd}}(X,f)$ is the $\mathbb{Q}$-linear dual of $H_n^{\mathrm{rd}}(X,f)$, that is:

$$H^n_{\mathrm{rd}}(X,f) = \mathrm{Hom}_{\mathbb{Q}}(H_n^{\mathrm{rd}}(X,f),\mathbb{Q}) = \underset{r\to+\infty}{\mathrm{colim}}\; H^n(X(\mathbb{C}), f^{-1}(S_r);\mathbb{Q}). \tag{1.1.1.2}$$

This cohomology theory for varieties with a potential enjoys many of the usual properties: the vector space $H^n_{\mathrm{rd}}(X,f)$ has finite dimension, depends functorially on $(X,f)$, satisfies a Künneth formula, fits into a Mayer–Vietoris long exact sequence, etc. Whenever $f$ is constant, the subspace $f^{-1}(S_r)$ is empty for big enough $r$, so one recovers the usual Betti cohomology of $X$.

As in the ordinary setting, rapid decay cohomology admits a purely algebraic description. Let $X$ be a smooth variety over a field $k$ of characteristic zero and $f\colon X\to\mathbb{A}^1$ a regular function. Let

$$\mathcal{E}^f = (\mathcal{O}_X, d_f)$$

denote the trivial rank one vector bundle on $X$ together with the integrable connection $d_f$ determined by $d_f(1) = -df$. The corresponding local system of analytic horizontal sections is the trivial local system generated by the exponential of $f$, which justifies the notation. However, being irregular singular at infinity, the connection $\mathcal{E}^f$ is non-trivial as long as $f$ is non-constant.

Let $DR(\mathcal{E}^f)$ be the de Rham complex of $\mathcal{E}^f$, namely

$$DR(\mathcal{E}^f)\colon\quad \mathcal{O}_X \xrightarrow{d_f} \Omega^1_X \xrightarrow{d_f} \cdots \xrightarrow{d_f} \Omega^{\dim X}_X,$$

where $d_f\colon \Omega^p_X\to\Omega^{p+1}_X$ is given by $d_f(\omega) = d\omega - df\wedge\omega$ on local sections $\omega$. By definition, the *de Rham cohomology* of the pair $(X,f)$ is the cohomology of this complex:

$$H^n_{\mathrm{dR}}(X,f) = H^n(X, DR(\mathcal{E}^f)). \tag{1.1.1.3}$$

As we shall see, using standard homological algebra, the above construction generalises to arbitrary $X$, not necessarily smooth, yielding another cohomology theory for varieties with potential. Again, the case where $f$ is constant gives back the usual de Rham cohomology of $X$.

1.1.2 (A comparison isomorphism). — Let $(X,f)$ be a smooth variety with potential defined over a subfield $k$ of $\mathbb{C}$. By a series of works starting from the aforementioned mentioned letter and continuing with Dimca–Saito [**DS93**], Sabbah [**Sab96**], Hien–Roucairol [**HR08**], and Hien [**Hie09**], there is a canonical comparison isomorphism

$$H^n_{\mathrm{dR}}(X,f)\otimes_k\mathbb{C} \xrightarrow{\sim} H^n_{\mathrm{rd}}(X,f)\otimes_{\mathbb{Q}}\mathbb{C},$$

which we shall most conveniently regard as a perfect pairing

$$H^n_{\mathrm{dR}}(X,f)\otimes H_n^{\mathrm{rd}}(X,f)\to\mathbb{C} \tag{1.1.2.1}$$

between de Rham cohomology and rapid decay homology.

Intuitively, elements of $H_n^{\mathrm{rd}}(X,f)$ are homology classes of topological cycles $\gamma$ on $X(\mathbb{C})$ which are not necessarily compact, but are only unbounded in the directions where $\mathrm{Re}(f)$ tends to infinity. More precisely, we view them as classes of compatible systems $\gamma = (\gamma_r)_{r\in\mathbb{R}}$ of compact cycles in $X(\mathbb{C})$ whose boundary $\partial\gamma_r$ is contained in $f^{-1}(S_r)$. Besides, when $X$ is affine, de Rham cohomology can be computed using global sections, so that elements of $H^n_{\mathrm{dR}}(X,f)$ are represented by $n$-forms $\omega$ on $X$. In this case, the pairing (1.1.2.1) sends $[\omega]\otimes[\gamma]$ to the integral

$$\int_\gamma e^{-f}\omega = \lim_{r\to+\infty}\int_{\gamma_r} e^{-f}\omega,$$

which is absolutely convergent since $e^{-f}$ *decays rapidly* in the directions where $\gamma$ is unbounded. The value of the integral is independent of the choice of representatives by Stokes' theorem: for example, two cohomologous forms will differ by $d_f(\eta)$ for some $\eta\in\Omega^{n-1}_X(X)$, and we have

$$\int_\gamma e^{-f}d_f(\eta) = \int_\gamma d(e^{-f}\eta) = \lim_{r\to+\infty}\int_{\partial\gamma_r} e^{-f}\eta = 0$$

because $\eta$ is algebraic and $e^{-f}$ goes to zero faster than any polynomial along the boundary of $\gamma_r$.

If the base field $k$ is further assumed to lie inside $\overline{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, one calls *exponential periods* the complex numbers arising as values of this pairing. Note that, when $f$ is a non-zero constant function, although we are dealing with the usual de Rham and singular cohomology of $X$, the comparison isomorphism is twisted by $e^{-f}$. For this reason, the exponentials of algebraic numbers are exponential periods associated with zero-dimensional varieties, and will play in what follows a similar role to algebraic numbers in the classical theory of periods.

1.1.3 (Examples). — We now present two more elaborated examples of exponential periods which appear, under various guises, in the work of Bloch–Esnault [**BE00**, §5], [**BE04**, p.360-361], Kontsevich–Zagier [**KZ01**, §4.3], Deligne [**DMR07**, p.115-128], Hien–Roucairol [**HR08**, p.529-530], and Bertrand [**Ber12**, §6].

Example 1.1.4. — Let $X=\operatorname{Spec} k[x]$ be the affine line and $f=a_nx^n+\ldots+a_0$ a polynomial of degree $n\geqslant 2$. The global de Rham complex of the connection $\mathcal{E}^f$ reads:

$$\begin{aligned} k[x] &\xrightarrow{d_f} k[x]dx \\ g &\longmapsto (g'-f'g)dx. \end{aligned}$$

Since $d_f$ is injective, the only non-trivial cohomology group is $H^1_{\mathrm{dR}}(X,f)=\operatorname{coker}(d_f)$. A basis is given by the differentials $dx, xdx,\ldots,x^{n-2}dx$. Indeed, these classes are linearly independent because the image of $d_f$ consists of elements of the form $hdx$ with $h$ of degree at least $n-1$. That they generate the whole cohomology can be seen by induction on noting that, for each $m\geqslant 0$, there is a polynomial $h_m$ of degree at most $n+m-2$ with

$$x^{n+m-1}dx - h_m dx = d_f\left(\tfrac{1}{na_n}x^m\right).$$

Let us now turn to rapid decay homology. The asymptotics of $\mathrm{Re}(f)$ being governed by the leading term of the polynomial, we may assume without loss of generality that $f=a_nx^n$ and write $a_n=ue^{i\alpha}$ with $u>0$ and $\alpha\in[0,2\pi)$. Given a real number $r>0$, the subspace $f^{-1}(S_r)\subseteq\mathbb{C}$ consists of the $n$ disjoint regions

$$\coprod_{j=0}^{n-1}\left\{ se^{i\theta}\ \middle|\ \frac{-\alpha+(2j-\frac12)\pi}{n}<\theta<\frac{-\alpha+(2j+\frac12)\pi}{n},\quad s\geqslant\left(\frac{r}{u\cos(\alpha+n\theta)}\right)^{\frac1n}\right\},$$

which are concentrated around the half-lines

$$\sigma_j = \left\{ se^{i\theta} \;\middle|\; \theta = \tfrac{-\alpha+2\pi j}{n},\ s \geqslant 0 \right\}, \qquad j = 0, \dots, n-1.$$

We orient each $\sigma_i$ from 0 to infinity. A basis of $H_1^{\mathrm{rd}}(X, f)$ is then given by the cycles

$$\gamma_i = \sigma_i - \sigma_0, \qquad i = 1, \dots, n-1.$$

Figure 1.1.1 illustrates the case of a polynomial of degree $n = 5$ whose leading term $a_n$ is a positive real number: the subspace $f^{-1}(S_r)$ is drawn in blue and the half-lines $\sigma_j$ in green.

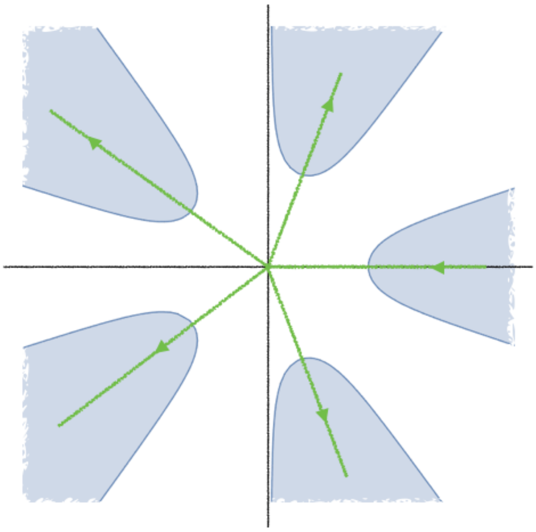

FIGURE 1.1.1. A basis of the rapid decay homology of a polynomial of degree 5 with positive leading term

With respect to these bases, the matrix of the period pairing (1.1.2.1) is

$$P = \left( \int_{\gamma_i} x^{j-1} e^{-f(x)} dx \right)_{i,j=1,\dots,n-1}.$$

Assuming that the base field $k$ is algebraic, the entries of $P$ are exponential periods. Let us see a few examples of familiar numbers which appear this way:

(i) Given a quadratic polynomial $f = ax^2 + bx + c$, the cohomology is one-dimensional. In this case, the cycle $-\gamma_1$ is the "rotated" real line $e^{-\frac{i\arg(a)}{2}}\mathbb{R}$, with its usual orientation, and one gets:

$$\int_{e^{-\frac{i\arg(a)}{2}}\mathbb{R}} e^{-ax^2-bx-c} dx = e^{\frac{b^2}{4a}-c} \sqrt{\frac{\pi}{a}}. \tag{1.1.4.1}$$

A particular case, for $f = x^2$, is the Gaussian integral

$$\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}, \tag{1.1.4.2}$$

which is not expected to be a period in the usual sense since, granted a theory of weights for periods, it would hint at the existence of a one-dimensional pure Hodge structure of weight one. We will prove in Section 12.2 that, assuming the analogue of the Grothendieck period conjecture for exponential motives, $\sqrt{\pi}$ is not a period of a usual motive.

(ii) More generally, consider the polynomial $f = x^n$ for $n \geqslant 2$. Set $\xi = e^{\frac{2\pi i}{n}}$ and let $\Gamma$ be the classical gamma function. Then the entries of $P$ are the exponential periods

$$\int_{\gamma_i} x^{j-1} e^{-x^n} dx = \frac{\xi^{ij}-1}{n} \int_0^{+\infty} x^{\frac{j}{n}-1} e^{-x} dx = \frac{\xi^{ij}-1}{n} \Gamma\left(\tfrac{j}{n}\right).$$

To get the special value of the gamma value alone, *i.e.* without the cyclotomic factor, it suffices to observe that the relation $\sum_{i=1}^{n-1} \xi^{ij} = -1$ yields

$$\Gamma\left(\tfrac{j}{n}\right) = \int_{-\gamma_1-\dots-\gamma_{n-1}} x^{j-1} e^{-x^n} dx. \tag{1.1.4.3}$$

Again, one does not expect *single* gamma values to be periods in the usual sense. However, we can obtain periods by taking suitable monomials in them.

Using geometric techniques inspired from the stationary phase formula—which will carry over to exponential motives—, Bloch and Esnault computed the determinant of the period matrix $P$ in [**BE00**, Prop. 5.4]:

$$\det P \sim_{k^\times} \sqrt{(-1)^{\frac{(n-1)(n-2)}{2}} s} \cdot \pi^{\frac{n-1}{2}} \cdot \exp(-\sum_{f'(\alpha)=0} f(\alpha)), \tag{1.1.4.4}$$

where $s = 1$ if $n$ is odd and $s = \frac{n}{2} a_n$ if $n$ is even. The symbol $\sim_{k^\times}$ means that the left and the right-hand side agree up to multiplication by an element of $k^\times$. Note the particular case (1.1.4.1).

Example 1.1.5. — Consider the torus $X = \operatorname{Spec} k[x, x^{-1}]$, together with the Laurent polynomial

$$f = -\tfrac{\lambda}{2}\left(x - \tfrac{1}{x}\right)$$

for some $\lambda \in k^\times$, which we view for the moment as a fixed parameter. Arguing as before, one sees that $\operatorname{coker}(d_f)$ is generated by $x^p dx$, for $p \in \mathbb{Z}$, modulo the relations

$$x^p dx + \tfrac{2p}{\lambda} x^{p-1} dx + x^{p-2} dx = 0.$$

It follows that the de Rham cohomology $H^1_{\mathrm{dR}}(X, f)$ is two-dimensional, a basis being given by the classes of the differentials $x^{-p-1}dx$ and $x^{-p}dx$ for any choice of an integer $p$.

On the rapid decay side, the subspace $f^{-1}(S_r) \subseteq \mathbb{C}^\times$ consists of two disjoint regions which are roughly a half-plane where $\operatorname{Re}(-\overline{\lambda} x)$ is large and the inversion with respect to the unit circle of the half-plane where $\operatorname{Re}(\lambda x)$ is large (see Figure 1.1.2 below). By passing to the limit $r \to +\infty$ in the long exact sequence of relative homology

$$\cdots \to H_1(f^{-1}(S_r), \mathbb{Q}) \to H_1(\mathbb{C}^\times, \mathbb{Q}) \to H_1(\mathbb{C}^\times, f^{-1}(S_r); \mathbb{Q}) \to H_0(f^{-1}(S_r), \mathbb{Q}) \to H_0(\mathbb{C}^\times, \mathbb{Q}) \to \cdots$$

one sees that $H_1^{\mathrm{rd}}(X, f)$ is two-dimensional and contains $H_1(\mathbb{C}^\times, \mathbb{Q})$. Therefore, a loop $\gamma_1$ winding once counterclockwise around 0 defines a class in rapid decay homology. To complete it to a basis, we consider any path joining the two connected components of $f^{-1}(S_r)$, for example the cycle $\gamma_2$ in $\mathbb{C}^\times$ consisting of the straight line from 0 (*not* included) to $\lambda$, the positive arc from $\lambda$ to $-\overline{\lambda}$ and the half-line from $-\overline{\lambda}$ towards $-\overline{\lambda}\infty$, as shown in Figure 1.1.2. Alternatively, we note that, on the vertical axis $\{x = it \mid t \in \mathbb{R}\}$, the real part of $f$ is given by $\mathrm{Re}(f) = \mathrm{Im}(\lambda)(t + \frac{1}{t})$, so, as long as $\lambda$ is not real, we can take the path $\gamma_2 : \mathbb{R}_{>0} \to \mathbb{C}^\times$ defined by

$$\gamma_2(t) = \begin{cases} it & \text{if } \mathrm{Im}(\lambda) > 0, \\ -it & \text{if } \mathrm{Im}(\lambda) < 0. \end{cases}$$

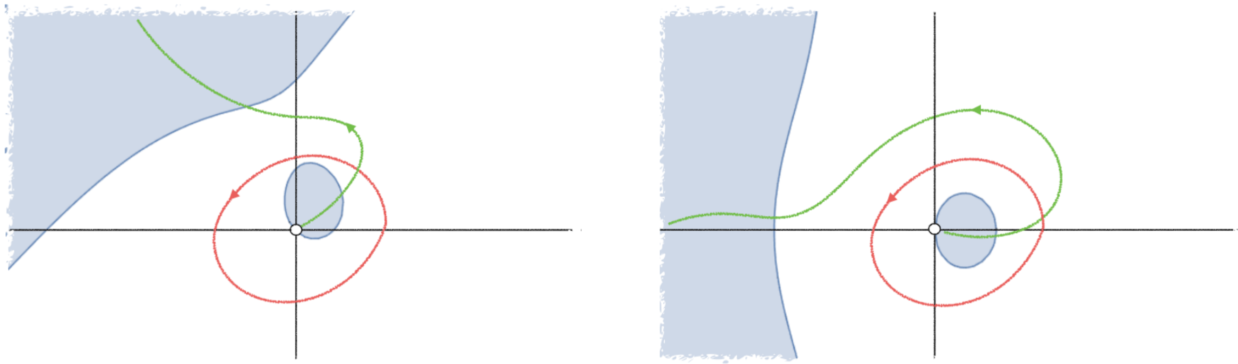

FIGURE 1.1.2. The subspaces $f^{-1}(S_r)$ and a basis of the rapid decay homology $H_1^{\mathrm{rd}}(X, f)$ when $\lambda = 1 + i$ (left) and $\lambda = 1$ (right)

Recall that, given an integer $n$, the Bessel function of the first kind of order $n$ is defined by

$$J_n(z) = \frac{1}{2\pi i} \int_{\gamma_1} e^{\frac{z}{2}(x - \frac{1}{x})} \frac{dx}{x^{n+1}}, \quad z \in \mathbb{C},$$

and the Bessel function of the third kind of order $n$ is defined by

$$H_n(z) = \frac{1}{\pi i} \int_{\gamma_2} e^{\frac{z}{2}(x - \frac{1}{x})} \frac{dx}{x^{n+1}}, \quad z \in \mathbb{C}^\times.$$

We adopt the conventions from [**Wat95**, 6.21]. The function $J_n(z)$ is entire whereas $H_n(z)$ is holomorphic on $\mathbb{C} \setminus i\mathbb{R}$ if the cycle $\gamma_2$ is given by the first description. The functions $J_n(z)$ and $H_n(z)$ are two linearly independent solutions of the second order linear differential equation

$$\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} + \left(1 - \frac{n^2}{z^2}\right) u = 0 \tag{1.1.5.1}$$

for an unknown function $u$ in one variable $z$. Observe that (1.1.5.1) has a regular singular point at $z = 0$ and an irregular singularity at infinity.

The matrix of the period pairing (1.1.2.1) with respect to the basis $x^{-n-1}dx$ and $x^{-n}dx$ of de Rham cohomology and $\gamma_1, \gamma_2$ of rapid decay homology reads

$$P = \begin{pmatrix} 2\pi i J_n(\lambda) & 2\pi i J_{n-1}(\lambda) \\ \pi i H_n(\lambda) & \pi i H_{n-1}(\lambda) \end{pmatrix}. \tag{1.1.5.2}$$

## 1.2. Exponential motives

1.2.1 (An abelian category after Nori). — According to the philosophy of motives, the existence of two cohomology theories for varieties with potential, as well as a comparison isomorphism between them, suggests looking for a universal cohomology with values in a tannakian category, from which any other reasonable cohomology theory would be obtained by composition with realisation functors. Such a category of *exponential motives* over a fixed subfield $k$ of $\mathbb{C}$ indeed exists, and we shall construct it using Nori's formalism [**Nor**].

Extending slightly the definition of rapid decay cohomology, we associate with a $k$-variety $X$, a closed subvariety $Y \subseteq X$, a regular function $f$ on $X$, and two integers $n$ and $i$ the vector space

$$\begin{aligned} \rho([X,Y,f,n,i]) &= H^n_{\mathrm{rd}}(X,Y,f)(i) \\ &= \operatorname*{colim}_{r\to+\infty} H^n(X(\mathbb{C}), Y(\mathbb{C}) \cup f^{-1}(S_r); \mathbb{Q})(i), \end{aligned} \tag{1.2.1.1}$$

where the twist $(i)$ means tensoring $-i$ times with the one-dimensional vector space $H^1(\mathbb{G}_m, \mathbb{Q})$. Note that we do not require any compatibility between the function and the subvariety.

Let us preliminarily write $\mathrm{Q}^{\exp}(k)$ for the category with objects the tuples $[X, Y, f, n, i]$ as above, and morphisms the maps of $k$-varieties compatible with the subvarieties and the functions in the obvious way. Then the assignment (1.2.1.1) defines a functor

$$\rho : \mathrm{Q}^{\exp}(k) \to \mathbf{Vec}_{\mathbb{Q}}. \tag{1.2.1.2}$$

The basic idea is to look at the endomorphism algebra of $\rho$, that is,

$$\mathrm{End}(\rho) = \{(e_q) \in \prod_{q \in Q^{\exp}(k)} \mathrm{End}(\rho(q)) \mid e_q \circ \rho(f) = \rho(f) \circ e_p \text{ for all } f : p \to q\}. \tag{1.2.1.3}$$

Filtering $\mathrm{Q}^{\exp}(k)$ by subcategories with a finite number of objects and morphisms, one sees that $\mathrm{End}(\rho)$ has a canonical structure of pro-algebra over $\mathbb{Q}$. Bearing this in mind, we tentatively define the category of *exponential motives* as

$$\mathbf{M}^{\exp}(k) = \left\{ \begin{array}{c} \text{finite-dimensional } \mathbb{Q}\text{-vector spaces endowed} \\ \text{with a continuous } \mathrm{End}(\rho)\text{-action} \end{array} \right\}. \tag{1.2.1.4}$$

The category $\mathbf{M}^{\exp}(k)$ is abelian, $\mathbb{Q}$-linear, and the functor $\rho$ lifts canonically to a functor $\tilde{\rho}\colon \mathrm{Q}^{\exp}(k) \to \mathbf{M}^{\exp}(k)$. The images of the objects of $Q^{\exp}(k)$ will be denoted by

$$H^n(X,Y,f)(i) = \tilde{\rho}([X,Y,f,n,i])$$

When $Y$ is empty or $i = 0$, we will usually drop them from the notation. In general, an exponential motive is a subquotient of a finite direct sum of objects of the form $H^n(X, Y, f)(i)$.

So far, there are no morphisms between objects of $\mathrm{Q}^{\exp}(k)$ with different $n$ or $i$. Yet, given a closed subvariety $Z$ of $Y$, there is a canonical morphism of vector spaces

$$\rho([Y, Z, f|_Y, n-1, i]) \to \rho([X, Y, f, n, i]) \tag{1.2.1.5}$$

which is induced, after passing to the limit, by the connecting morphism in the long exact sequence for the closed immersions $Z \cup f^{-1}(S_r) \subseteq Y \cup f^{-1}(S_r) \subseteq X$. We would like to lift this morphism to the category $\mathbf{M}^{\exp}(k)$. To achieve this, we simply add to $\mathrm{Q}^{\exp}(k)$ an artificial morphism

$$[X, Y, f, n, i] \to [Y, Z, f|_Y, n-1, i],$$

and declare its image under $\rho$ to be (1.2.1.5). As we do not specify any composition law for the new morphisms, $\mathrm{Q}^{\exp}(k)$ ceases to be a category, and is now only a *quiver* (or a *diagram* in Nori's terminology). By that, we understand a collection of objects, morphisms with source and target, and specified identity morphisms (see Section 4.1 for a reminder).

The definitions (1.2.1.3) and (1.2.1.4) are still meaningful, and now the morphisms (1.2.1.5) obviously lift to $\mathbf{M}^{\exp}(k)$. After introducing a second class of extra morphisms to $\mathrm{Q}^{\exp}(k)$, which relate objects having different twists, we arrive at our final definition of the quiver $\mathrm{Q}^{\exp}(k)$ and the category $\mathbf{M}^{\exp}(k)$. We will call *Betti realisation* the forgetful functor

$$R_B \colon \mathbf{M}^{\exp}(k) \longrightarrow \mathbf{Vec}_{\mathbb{Q}}. \tag{1.2.1.6}$$

Adapted to our context, Nori's main theorem about the categories associated to quiver representations [**Nor, HMS17**] says that $\mathbf{M}^{\exp}(k)$ is universal for all cohomology theories which are comparable to rapid decay cohomology. More precisely, one has the following result:

THEOREM 1.2.2 (Nori). — *Let $F$ be a field of characteristic zero and $\mathbf{A}$ an abelian, $F$-linear category together with an exact, $F$-linear, faithful functor $\mathbf{A} \to \mathbf{Vec}_F$. Let $h\colon \mathrm{Q}^{\exp}(k) \to \mathbf{A}$ be a functor, and suppose that natural isomorphisms of vector spaces*

$$h([X, Y, f, n, i]) \simeq \rho([X, Y, f, n, i]) \otimes_{\mathbb{Q}} F$$

*are given for each object $[X, Y, f, n, i]$. Then there exists a unique functor, up to isomorphism, $R_{\mathbf{A}} : \mathbf{M}^{\exp}(k) \to \mathbf{A}$ such that $h$ is the composite of $R_{\mathbf{A}}$ and the canonical lift $\widetilde{\rho}\colon \mathrm{Q}^{\exp}(k) \to \mathbf{M}^{\exp}(k)$.*

This universal property will be used to construct other realisation functors. Important examples are the period and the perverse realisations, which we now discuss.

1.2.3 (The period realisation). — A *period structure* over $k$ is a triple $(V, W, \alpha)$ consisting of a $\mathbb{Q}$-vector space $V$, a $k$-vector space $W$, and an isomorphism $\alpha\colon V \otimes_{\mathbb{Q}} \mathbb{C} \to W \otimes_k \mathbb{C}$ of complex vector spaces. Together with the obvious morphisms, period structures form an abelian $\mathbb{Q}$-linear category $\mathbf{PS}(k)$. There is a forgetful functor $\mathbf{PS}(k) \to \mathbf{Vec}_{\mathbb{Q}}$ sending $(V, W, \alpha)$ to $V$.

Extending the definition of de Rham cohomology and the comparison isomorphism from 1.1.1 and 1.1.2 to the relative setting and singular varieties, one obtains a functor $\mathrm{Q}^{\exp}(k) \to \mathbf{PS}(k)$, whose composition with the forgetful functor is nothing else but $\rho$. Therefore, Nori's Theorem 1.2.2 yields an exact and faithful functor

$$R_{\mathbf{P}} : \mathbf{M}^{\exp}(k) \to \mathbf{PS}(k),$$

which we call the *period realisation.* Composing with the functor $\mathbf{PS}(k) \to \mathbf{Vec}_k$ sending $(V, W, \alpha)$ to $W$, we obtain the *de Rham realisation*

$$R_{\mathrm{dR}} \colon \mathbf{M}^{\exp}(k) \longrightarrow \mathbf{Vec}_k.$$

1.2.4 (The perverse realisation). — We now turn to another realisation functor which takes values in a subcategory of perverse sheaves with rational coefficients on the complex affine line. Recall that, given two objects $A$ and $B$ of the derived category of constructible sheaves of $\mathbb{Q}$-vector spaces on $\mathbb{A}^1(\mathbb{C})$, one defines their *additive convolution* by

$$A * B = \mathrm{Rsum}_*(\mathrm{pr}_1^* A \otimes \mathrm{pr}_2^* B),$$

where $\mathrm{sum}\colon \mathbb{A}^2 \to \mathbb{A}^1$ is the summation map, and $\mathrm{pr}_i\colon \mathbb{A}^2 \to \mathbb{A}^1$ the projections onto the two factors. Even if we start with two perverse sheaves, their additive convolution fails to be perverse in general. To remedy this, we will restrict to the full subcategory $\mathbf{Perv}_0$ of $\mathbb{Q}$-perverse sheaves on $\mathbb{A}^1(\mathbb{C})$ consisting of those objects $C$ without global cohomology, *i.e.* such that $R\pi_* C = 0$ for $\pi$ the structure morphism of $\mathbb{A}^1$. A typical object of this category is $E(0) = j_! j^* \underline{\mathbb{Q}}[1]$, where $j\colon \mathbb{G}_m \hookrightarrow \mathbb{A}^1$ stands for the natural inclusion. Indeed, we shall see that all the objects of $\mathbf{Perv}_0$ are of the form $F[1]$ for some constructible sheaf of $\mathbb{Q}$-vector spaces $F$ satisfying $H^*(\mathbb{A}^1(\mathbb{C}), F) = 0$. This enables us to define the "nearby fibre at infinity" $\Psi_\infty\colon \mathbf{Perv}_0 \to \mathbf{Vec}_{\mathbb{Q}}$ as

$$\Psi_\infty(F[1]) = \lim_{r \to +\infty} F(S_r).$$

Besides, the inclusion of $\mathbf{Perv}_0$ into $\mathbf{Perv}$ admits a left adjoint $\Pi\colon \mathbf{Perv} \to \mathbf{Perv}_0$ which is given by additive convolution with the object $E(0)$, that is, $\Pi(C) = C * E(0)$.

For a variety $X$ and a closed subvariety $Y \subseteq X$, let $\beta\colon X \setminus Y \hookrightarrow X$ be the inclusion of the complement and $\underline{\mathbb{Q}}_{[X,Y]} = \beta_! \beta^* \underline{\mathbb{Q}}$ the sheaf computing the relative cohomology of the pair $(X(\mathbb{C}), Y(\mathbb{C}))$. We define a functor $\mathrm{Q}^{\exp}(k) \to \mathbf{Perv}_0$ by assigning to $[X, Y, f, n, i]$ the perverse sheaf

$$\Pi({}^p\mathcal{H}^n(Rf_* \underline{\mathbb{Q}}_{[X,Y]}))(i),$$

where ${}^p\mathcal{H}^n$ stands for the cohomology with respect to the $t$-structure defining $\mathbf{Perv}$ inside the derived category of constructible sheaves. As we shall prove in 3.2, the composition of this functor with $\Psi_\infty$ gives back the rapid decay cohomology. Invoking the universal property again, this yields the *perverse realisation*

$$R_{\mathbf{Perv}}\colon \mathbf{M}^{\exp}(\mathbf{k}) \longrightarrow \mathbf{Perv_0}.$$

1.2.5 (The tensor structure). — Given two pairs $(X_1, f_1)$ and $(X_2, f_2)$ of varieties with potential, the cartesian product $X_1 \times X_2$ is equipped with the *Thom–Sebastiani sum*

$$(f_1 \boxplus f_2)(x_1, x_2) = f_1(x_1) + f_2(x_2). \tag{1.2.5.1}$$

There is a cup-product in rapid decay cohomology

$$H^{n_1}_{\mathrm{rd}}(X_1, Y_1, f_1) \otimes H^{n_2}_{\mathrm{rd}}(X_2, Y_2, f_2) \longrightarrow H^{n_1+n_2}_{\mathrm{rd}}(X_1 \times X_2, (Y_1 \times X_2) \cup (X_1 \times Y_2), f_1 \boxplus f_2)$$

which induces an isomorphism of $\mathbb{Q}$-vector spaces (*Künneth formula*):

$$\bigoplus_{a+b=n} H^a_{\mathrm{rd}}(X_1, Y_1, f_1) \otimes H^b_{\mathrm{rd}}(X_2, Y_2, f_2) \simeq H^n_{\mathrm{rd}}(X_1 \times X_2, (Y_1 \times X_2) \cup (X_1 \times Y_2), f_1 \boxplus f_2).$$

The technical heart of this work is the following theorem:

THEOREM 1.2.6 (cf. Theorem 4.4.1). — *There exists a unique monoidal structure on* $\mathbf{M}^{\exp}(k)$ *which is compatible with the Betti realisation* $R_B\colon \mathbf{M}^{\exp}(k) \to \mathbf{Vec}_{\mathbb{Q}}$ *and with cup-products. With respect to this monoidal structure,* $\mathbf{M}^{\exp}(k)$ *is a neutral tannakian category with* $R_B$ *as fibre functor.*

The difficulty of constructing the tensor product stems from the fact that the obvious rule

$$[X_1, Y_1, f_1, n_1, i_1] \otimes [X_2, Y_2, f_2, n_2, i_2] = [X_1 \times X_2, (Y_1 \times X_2) \cup (X_1 \times Y_2), f_1 \boxplus f_2, n_1 + n_2, i_1 + i_2]$$

is not compatible with the Künneth formula unless the rapid decay cohomology of the triples $(X_i, Y_i, f_i)$ is concentrated in a single degree. As for usual Nori motives, the problem is solved by showing that every object admits a "cellular filtration". More precisely, the key ingredient is the following statement, which—thanks to the perverse realisation—follows from Beilinson's most general form of the basic lemma.

THEOREM 1.2.7 (Exponential basic lemma, cf. Corollary 3.3.3). — *Let* $X$ *be an affine variety of dimension* $d$*, together with a regular function* $f\colon X \to \mathbb{A}$*, and let* $Y \subsetneq X$ *be a closed subvariety, There exists a closed subvariety* $Z \subseteq X$ *of dimension at most* $d-1$ *and containing* $Y$ *such that* $H^i(X, Z, f) = 0$ *for all* $i \neq d$*.*

Once we have the tensor product at our disposal, many relations between exponential periods can be proved to be of motivic origin. For instance, the value (1.1.4.2) of the Gaussian integral reflects the isomorphism of motives

$$H^1(\mathbb{A}^1, x^2)^{\otimes 2} = H^1(\{x^2 + y^2 = 1\})$$

which will be established in Section 12.2.

1.2.8 (Relation with usual Nori motives). — Nori's category of (non-effective cohomological) mixed motives over $k$ is related to our construction as follows. Let $\mathrm{Q}(k)$ be the full subquiver of $\mathrm{Q}^{\exp}(k)$ consisting of those tuples $[X, Y, f, n, i]$ with $f = 0$. The restriction of the representation $\rho$ to this subquiver is nothing other than the usual Betti cohomology of the pair $(X(\mathbb{C}), Y(\mathbb{C}))$. Nori's category of mixed motives $\mathbf{M}(k)$ is the category of finite-dimensional $\mathbb{Q}$-vector spaces equipped with a continuous $\mathrm{End}(\rho|_{\mathrm{Q}(k)})$-action. From the inclusion $\mathrm{Q}(k) \to \mathrm{Q}^{\exp}(k)$, one obtains a restriction homomorphism $\mathrm{End}(\rho) \to \mathrm{End}(\rho|_{\mathrm{Q}(k)})$, hence a canonical functor $\mathbf{M}(k) \to \mathbf{M}^{\exp}(k)$ which, by the general formalism, is faithful and exact.

THEOREM 1.2.9 (cf. Theorem 5.1.1). — *The functor* $\mathbf{M}(k) \longrightarrow \mathbf{M}^{\exp}(k)$ *is full.*

This enables us to identify Nori's usual motives with a subcategory of exponential motives. However, the image of $\mathbf{M}(k)$ in $\mathbf{M}^{\exp}(k)$ is *not* stable under extension. In Chapter 12, we shall construct an extension of $\mathbb{Q}(-1)$ by $\mathbb{Q}(0)$ whose period matrix is given by

$$\begin{pmatrix} 1 & \gamma \\ 0 & 2\pi i \end{pmatrix},$$

where $\gamma = \lim_{n\to\infty}(\sum_{k=1}^{n} \frac{1}{k} - \log(n))$ denotes the Euler-Mascheroni constant.

## 1.3. The motivic exponential Galois group

By the fundamental theorem of tannakian categories, $\mathbf{M}^{\exp}(k)$ is equivalent to the category of representations of an affine group scheme $G^{\exp}(k)$ over $\mathbb{Q}$, which will be called the *motivic exponential Galois group*. A formal consequence of the construction of the tannakian category $\mathbf{M}^{\exp}(k)$ and the realisations functors will be the following:

Proposition 1.3.1 (cf. Proposition 8.4.1). — *The scheme of tensor isomorphisms*

$$\underline{\mathrm{Isom}}^{\otimes}(R_{\mathrm{dR}}, R_B)$$

*is a torsor under the motivic exponential Galois group.*

Given an exponential motive $M$, one can look at the smallest tannakian subcategory $\langle M\rangle^{\otimes}$ of $\mathbf{M}^{\exp}(k)$ containing $M$. Invoking again the general formalism, $\langle M\rangle^{\otimes}$ is equivalent to $\mathbf{Rep}(G_M)$ for a linear algebraic group $G_M \subseteq \mathrm{GL}(R_B(M))$ which we call the *Galois group* of $M$. It follows from Proposition 1.3.1 that, when $k$ is a number field, the dimension of $G_M$ is a un upper bound for the transcendence degree of the field generated by the periods of $M$. Indeed, one conjectures:

Conjecture 1.3.2 (Exponential period conjecture, cf. Conjecture 8.2.3). — *Given an exponential motive $M$ over a number field, one has*

$$\mathrm{trdeg}\, \overline{\mathbb{Q}}(\textit{periods of } M) = \dim G_M.$$

A number of classical results and conjectures in transcendence theory may be seen as instances of this equality. For example, we will show in Section 12.1 that the Lindemann-Weierstrass theorem (given $\mathbb{Q}$-linearly independent algebraic numbers $\alpha_1, \dots, \alpha_n$, their exponentials $e^{\alpha_1}, \dots, e^{\alpha_n}$ are algebraically independent) is the exponential period conjecture for the motive

$$M = \bigoplus_{i=1}^{n} H^0(\mathrm{Spec}\, k, -\alpha_i),$$

where $k$ denotes the number field generated by $\alpha_1, \dots, \alpha_n$.

1.3.3 (Gamma motives and the abelianisation of the exponential Galois group). — For each integer $n \geqslant 2$, consider the following exponential motive over $\mathbb{Q}$:

$$M_n = H^1(\mathbb{A}^1, x^n). \tag{1.3.3.1}$$

By Example 1.1.4, all the values of the gamma function at rational numbers of denominator $n$ are periods of $M_n$, so it makes sense to call (1.3.3.1) a *gamma motive*. Avatars of the $M_n$ already appeared in the work of Anderson [**An86**] under the name of *ulterior motives*. The rationale behind his choice of terminology was that, while $M_n$ are "not themselves motives, motives may be constructed from them via the operations of linear algebra" (*loc.cit.*, p.154). As a striking illustration, he showed that, for all $m \geqslant 2$, the tensor product $M_n^{\otimes m}$ contains a submotive isomorphic to the primitive motive of Fermat hypersurface $X = \{x_1^n + \ldots + x_m^n = 0\} \subseteq \mathbb{P}^{m-1}$. We shall recover this fact in a very natural way in Chapter 13, cf. Proposition 13.1.3.

Conjecture 1.3.4 (Lang). — *Let $n \geqslant 3$ be an integer. The transcendence degree of the field generated over $\overline{\mathbb{Q}}$ by the gamma values $\Gamma(\frac{a}{n})$, for $a = 1, \ldots, n-1$, is equal to $1 + \frac{\varphi(n)}{2}$.*

At the time of writing, the conjecture is only known for $n = 3, 4, 6$, as a corollary of Chudnovsky's theorem that the transcendence degree of the field of periods of an elliptic curve over $\overline{\mathbb{Q}}$ is at least 2 and the Chowla-Selberg formula [**Chu84**].

As observed by André [**And04**, 24.6], this conjecture follows from Grothendieck's period conjecture, although in a rather indirect way which requires to know that periods of abelian varieties with complex multiplication by a cyclotomic field can be expressed in terms of gamma values. We shall prove that the Galois group of the motive $M_n$ fits into an exact sequence

$$0 \to \mu_n \to G_{M_n} \longrightarrow \mathcal{S}^{\mathbb{Q}(\mu_n)} \to 0,$$

where $\mathcal{S}^{\mathbb{Q}(\mu_n)}$ stands for the Serre torus of the cyclotomic field $\mathbb{Q}(\mu_n)$. This implies that $G_{M_n}$ has dimension $1 + \frac{\varphi(n)}{2}$ and enables us to see Lang's conjecture as an instance of Conjecture 1.3.2.

## 1.4. Outline

Briefly, the text is organized as follows. We refer the reader to the introductions of each chapter for a more precise description.

Section 2 contains some preliminaries about perverse sheaves that will be used in the sequel. The main result is that the category $\mathbf{Perv}_0$ is tannakian with respect to the monoidal structure given by additive convolution and the nearby fibre at infinity functor. We then discuss another fibre functor, given by the total vanishing cycles. A careful study of the local monodromies of the additive convolution allows one to see the compatibility with the tensor structures as a reformulation of the Thom–Sebastiani theorem.

In Chapter 3, we study the basic properties of rapid decay cohomology. Besides the elementary definition, we give two alternative descriptions. The first one, as the nearby fibre at infinity of a perverse sheaf, is used to obtain the exponential basic lemma. The second one, in terms of the oriented real blow-up, will play a pivotal role in the proof of the comparison isomorphism.

Chapter 4 is the technical core of this work. After some preliminaries about Nori's formalism, we define $\mathbf{M}^{\exp}(k)$ as an abelian category and prove that usual Nori motives form a full subcategory. We then move to the construction of the tensor product. In the last sections, we show that the Gysin morphism is motivic and we complete the proof that $\mathbf{M}^{\exp}(k)$ is tannakian.

Chapter 7 is devoted to the comparison between rapid decay and de Rham cohomology. Revisiting work of Hien and Roucairol, we prove a Poincaré lemma for the moderate growth twisted de Rham complex and use it to construct the period pairing.

Chapter 8 exploits the results of the previous chapter to obtain the period realisation functor. We then discuss a number of related topics, especially the notion of motivic exponential period and the coaction of the Galois group.

In Chapter 9

Chapter 11 deals with exponential Hodge theory. We upgrade the perverse realisation to a Hodge realisation with values in a subcategory of mixed Hodge modules on the affine line. We then prove that the weight filtration is motivic and discuss briefly the irregular Hodge filtration.

In Chapter 12, we present a collection of examples of exponential motives and compute their periods and Galois groups. These include exponentials of algebraic numbers, the motive $\mathbb{Q}(\frac{1}{2})$, special values of the Bessel functions and the Euler-Mascheroni constant.

Finally, in Chapter 13 we examine the gamma motives $M_n$. We compute their Galois groups and show that their dimensions are in accordance with Lang's conjecture. From this we obtain a conjectural description of the abelianisation of the exponential motivic Galois group.

1.4.1 (Notation and conventions). — Throughout, $k$ denotes a subfield of $\mathbb{C}$. By a variety over $k$ we mean a quasi-projective separated scheme of finite type over $k$. We shall call normal crossing divisor what is usually called a simple or strict normal crossing divisor, *i.e.* the irreducible components are smooth. Although this assumption is not indispensable for all constructions, there will be no lost in making it. Given a variety $X$, a closed subvariety $Y \subseteq X$ and a constructible sheaf $F$ on $X$, we set $F_{[X,Y]} = \beta_! \beta^* F$ where $\beta\colon X \setminus Y \hookrightarrow X$ is the inclusion of the complement.

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CHAPTER 2

# The category $\mathbf{Perv}_0$

In this chapter, we introduce the category $\mathbf{Perv}_0$ following Katz [**Kat90**, 12.6] and Kontsevich and Soibelman [**KS11**, 4.2]. It is the full subcategory of perverse sheaves with rational coefficients on the complex affine line consisting of those objects with no global cohomology. After some preliminaries about constructible and perverse sheaves, we study the basic properties of $\mathbf{Perv}_0$, the main result being that, together with additive convolution and a "nearby fibre at infinity" functor, it has the structure of a tannakian category. This category will play a pivotal role in the description of rapid decay cohomology and the proof of the exponential basic lemma in Chapter 3. Later on, it is indispensable for the construction of the Hodge realisation functor.

## 2.1. Prolegomena on perverse sheaves

In this section, we collect a few basic definitions and facts about perverse sheaves which will be used in the sequel. Our standard references are [**BBD82**], [**KS90**], [**Sch03**] or [**Dim04**]. We convene that "sheaf" means "sheaf of $\mathbb{Q}$-vector spaces" unless otherwise indicated.

2.1.1 (Constructible sheaves and the six functors formalism). — Given a variety $X$ over a subfield $k$ of $\mathbb{C}$, we let $\mathrm{Sh}(X)$ denote the abelian category of sheaves on the topological space $X(\mathbb{C})$. We will denote the derived category of $\mathrm{Sh}(X)$ by $D(X)$, and its bounded derived category by $D^b(X)$. We say that a sheaf $F$ in $\mathrm{Sh}(X)$ is *constructible* if there exist closed subvarieties

$$\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X_r = X$$

such that the restriction of $F$ to $X_p(\mathbb{C}) \setminus X_{p-1}(\mathbb{C})$ is a local system of finite rank for $p = 0, \dots, r$. If two terms in a short exact sequence of sheaves on $X$ are constructible, then so is the third one. Constructible sheaves form thus an abelian subcategory of $\mathrm{Sh}(X)$, which is moreover stable under tensor products and internal $\mathcal{H}om$.

Definition 2.1.2. — The *bounded derived category of constructible sheaves* $D^b_c(X)$ is the full subcategory of $D^b(X)$ consisting of those complexes $C$ whose homology sheaves $\mathcal{H}^q(C)$ are constructible for all integers $q$. Slightly abusively, we will also call constructible sheaf an object $C$ of $D^b_c(X)$ such that $\mathcal{H}^q(C) = 0$ unless $q = 0$.

Remark 2.1.3. — The terminology is not completely abusive. Writing $D^b(\mathrm{Constr}(X))$ for the bounded derived category of the abelian category constructible sheaves on $X$, the obvious functor

$$D^b(\mathrm{Constr}(X)) \to D^b_c(X)$$

is an equivalence of categories by [**No00**, Theorem 3(b)].

2.1.4. — Associated with each morphism $f\colon X \to Y$ of algebraic varieties, there are functors

| | | |
|---|---|---|
| $f^*\colon$ | $\mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ | inverse image |
| $f_*\colon$ | $\mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ | direct image |
| $f_!\colon$ | $\mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ | direct image with compact support. |

The inverse image functor is exact, whereas the direct image functors are only left exact. Taking their derived functors yields $f^*\colon D(Y) \to D(X)$ and $Rf_*, Rf_!\colon D(X) \to D(Y)$. The functors $f^*$ and $Rf_*$ are adjoint to each other, so there is a natural adjunction isomorphism

$$\mathrm{Hom}_{D(Y)}(A, Rf_*B) = \mathrm{Hom}_{D(X)}(f^*A, B)$$

for all objects $A$ of $D(Y)$ and $B$ of $D(X)$. It is a non-trivial result that the functor $Rf_!$ admits a right adjoint $f^!\colon D(Y) \to D(X)$, so there is a natural adjunction isomorphism

$$\mathrm{Hom}_{D(Y)}(Rf_!B, A) = \mathrm{Hom}_{D(X)}(B, f^!A)$$

for all objects $A$ of $D(Y)$ and $B$ of $D(X)$. This adjoint $f^!$ only exists at the level of derived categories, the functor $f_!$ between categories of sheaves has in general no right adjoint. The situation is summarised in the following diagram:

$$X \xrightarrow{f} Y \qquad \rightsquigarrow \qquad D(X) \underset{f^*}{\overset{Rf_*}{\rightleftarrows}} D(Y) \underset{Rf_!}{\overset{f^!}{\rightleftarrows}} D(X),$$

where functors on top are right adjoint to functors below.

The functor *sheaf of homomorphisms* associating with sheaves $F$ and $G$ on $X$ the sheaf $\mathcal{H}om(F, G)$ on $X$ can be derived as a left exact functor in $G$, giving rise to the functor

$$R\mathcal{H}om\colon D(X)^{\mathrm{op}} \times D(X) \longrightarrow D(X).$$

Since we will be only considering sheaves of vector spaces, the functor associating with sheaves $F$ and $G$ on $X$ the tensor product sheaf $F \otimes G$ is exact in both variables and does not need to be derived. Given objects $A, B, C$ of $D(X)$, the usual adjunction formula holds, in that there is a canonical isomorphism

$$R\mathcal{H}om(A \otimes B, C) = R\mathcal{H}om(A, R\mathcal{H}om(B, C))$$

in $D(X)$ which is natural in the three arguments. The functors $Rf_*, f^*, Rf_!, f^!, \otimes, R\mathcal{H}om$ are usually referred to as the *six operations*.

Theorem 2.1.5 (Verdier's constructibility theorem). — *The six operations preserve the derived categories of constructible sheaves.*

To the authors knowledge, Verdier never stated this theorem explicitly. Stability under $f^*$, $\otimes$ and $R\mathcal{H}om$ is straightforward. As explained in [**BBD82**, 2.1.13 and 2.2.1], the statement that $Rf_*, Rf_!$ and $f^!$ preserve constructibility follows formally from the fact that every stratification of an algebraic variety can be refined to a *Whitney stratification*, which is proven by Verdier in [**Ver76**, Théorème 2.2]. One can also prove it by induction on the dimension of supports and using the fact that, for every morphism of complex algebraic varieties $f\colon X \to Y$, there exists a non-empty Zariski open subset $U \subseteq Y$ such that $f^{-1}(U) \to U$ is a fibre bundle for the complex topology. This statement appears as Corollaire 5.1 in [**Ver76**], and can also be proved using resolution of singularities and Ehresmann's fibration theorem. A quite different approach is taken by Nori in [**No00**, Theorem 4], where he shows that one can compute $Rf_*$ using a resolution by constructible sheaves. Hence, in order to show that $Rf_*$ preserves constructibility it is enough to show that $f_*$ does so, which is not difficult. A proof of Verdier's constructibility theorem in a more general context is given in chapter 4 of [**Sch03**].

2.1.6. — Let $f\colon X \to Y$ be a morphism of algebraic varieties. In special cases, depending on the quality of $f$, direct and inverse image functors between derived categories of constructible sheaves satisfy the following relations. We collect them here pour mémoire:

(1) If $f$ is *proper*, then $Rf_* = Rf_!$.
(2) If $f$ is a *smooth* morphism of relative dimension $d$, then $f^! = f^*[2d]$.
(3) If $f$ is a *closed immersion*, then $f_*$ is exact.
(4) If $f$ is an *open immersion*, then $f_!$ is exact.

An example for (2) is given in 2.1.10 below.

2.1.7 (Base change theorems). — Consider a cartesian square of complex algebraic varieties

$$\begin{array}{ccc} X' & \xrightarrow{g_X} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g_Y} & Y, \end{array} \tag{2.1.7.1}$$

*i.e.* $X'$ is the fibre product of $X$ and $Y'$ over $Y$. For every sheaf $F$ on $X$, or more generally for every object $A$ of $D(X)$, there is a canonical, natural morphism

$$g_Y^* Rf_* A \to Rf'_* g_X^* A \tag{2.1.7.2}$$

of sheaves on $Y'$ called *base change morphism*. In general, (2.1.7.2) is not an isomorphism. In two important geometric cases, the base change morphism is an isomorphism. The *proper base change theorem* states that if $f$ is a proper morphism, then (2.1.7.2) is an isomorphism for all objects $A$ of $D(X)$. In particular, there is an isomorphism

$$g_Y^* Rf_! A \xrightarrow{\cong} Rf'_! g_X^* A \tag{2.1.7.3}$$

without any condition on $f$. The *smooth base change theorem* states that if $g_Y$ is smooth, then (2.1.7.2) is an isomorphism for all objects $A$ of the derived category of constructible sheaves $D^b_c(X)$. This can be deduced from point (2) of 2.1.6 and base change for the exceptional inverse image, which gives a canonical, natural isomorphism

$$g_Y^! Rf_* A \xrightarrow{\cong} Rf'_* g_X^! A \tag{2.1.7.4}$$

without any condition on $g_Y$. Proofs can be found in [**KS90**], see Proposition 2.5.11 for proper base change, and Proposition 3.1.9 for smooth base change.

2.1.8. — Let $X$ be a variety over $k$ and $\pi\colon X \to \operatorname{Spec}(k)$ the structure morphism. The *dualising complex* of $X$ (sometimes also called *dualising sheaf*, although it is not really a sheaf) is the object

$$\omega_X = \pi^! \mathbb{Q}$$

in the derived category of constructible sheaves $D^b_c(X)$. More generally, the *relative dualising complex* for a morphism $f\colon X \to Y$ is defined as $\omega_{X/Y} = f^! \mathbb{Q}_Y$. One then defines the *Verdier dual* of an object $A$ of $D^b_c(X)$ as

$$\mathbb{D}(A) = R\mathcal{H}om(A, \omega_X).$$

Theorem 2.1.9 (Local Verdier Duality). — *Given a morphism $f\colon X \to Y$ of algebraic varieties and objects $A$ of $D^b_c(X)$ and $B$ of $D^b_c(Y)$, there is a natural isomorphism*

$$R\mathcal{H}om(Rf_! A, B) \cong Rf_* R\mathcal{H}om(A, f^! B) \tag{2.1.9.1}$$

*in the category $D^b_c(X)$. In particular, there are natural isomorphisms $\mathbb{D}(Rf_! A) \cong Rf_* \mathbb{D}(A)$ and $\mathbb{D}(\mathbb{D}(A)) \cong A$.*

References are [**KS90**, Proposition 3.1.10] or [**Dim04**, Theorem 3.2.3]. Taking global sections on both sides of (2.1.9.1) yields the global form of Verdier's Duality Theorem, namely:

2.1.10. — The dualising complex $\omega_X$ on $X$ has the following explicit description. For any open set $U \subseteq X(\mathbb{C})$, let $\dot{U} = U \cup \{\cdot\}$ be the one point compactification of $U$, and let

$$C_*(\dot{U}, \{\cdot\}) = [\cdots \to C_2(\dot{U}, \{\cdot\}) \to C_1(\dot{U}, \{\cdot\}) \to C_0(\dot{U}, \{\cdot\})]$$

be the singular chain complex of the pair $(U, \{\cdot\})$ with $\mathbb{Q}$-coefficients. We view this as a complex concentrated in degrees $\leqslant 0$. For any inclusion of open sets $V \subseteq U$ there is a canonical map $\dot{U} \to \dot{V}$ contracting $U \setminus V$ to $\cdot \in V$. This map yields a morphism of complexes $C_*(\dot{U}, \{\cdot\}) \to C_*(\dot{V}, \{\cdot\})$. The dualising complex is the complex of sheaves associated with the presheaves $U \longmapsto C_*(\dot{U}, \{\cdot\})$. In particular, $\mathcal{H}^p(\omega_X)$ is the sheaf associated with the presheaf given by reduced singular homology $U \longmapsto \widetilde{H}_p(\dot{U})$. The easiest example where this recipe for computing the dualising complex yields a concrete description is the case where $X$ is smooth of dimension $d$. In that case, $X(\mathbb{C})$ is locally homeomorphic to an open ball of real dimension $2d$, so every point $x \in X(\mathbb{C})$ has a fundamental system of open neighbourhoods $U$ for which $\dot{U}$ is homeomorphic to a sphere of dimension $2d$. It follows that the dualising complex is isomorphic to $\mathbb{Q}_X[2d]$.

To see what Verdier's local duality theorem has to do with more classical duality theorems, consider a smooth variety $X$, and take for $f$ the structure morphism. Choose for $A$ the constant sheaf $\mathbb{Q}_X$ on $X$, and for $B$ the sheaf $\mathbb{Q}$ on the point. The complex $Rf_!A$ computes the cohomology with compact support $H^p_c(X(\mathbb{C}), \mathbb{Q})$ of $X$, whereas $R\mathcal{H}om(Rf_!A, B)$ computes its linear dual $H^p_c(X(\mathbb{C}), \mathbb{Q})^\vee$. The sheaf $R\mathcal{H}om(A, f^!B)$ is the dualising sheaf $\omega_X = \mathbb{Q}_X[2d]$, hence $Rf_*R\mathcal{H}om(A, f^!B)$ computes the homology of $X$. Bookkeeping the shifting, the canonical isomorphism in Verdier's duality theorem boils down to the classical Poincaré duality pairing

$$H^p_c(X(\mathbb{C}), \mathbb{Q}) \otimes H^{2d-p}(X(\mathbb{C}), \mathbb{Q}) \to \mathbb{Q}$$

between cohomology and cohomology with compact support.

Theorem 2.1.11 (Artin's Vanishing Theorem). — *Let $X$ be an affine variety over $k$, and let $F$ be a constructible sheaf on $X$. Then $H^q(X, F) = 0$ for all $q > \dim X$.*

The original reference is Artin's Exposé XIV in SGA 4 [**Art73**]. An analytic proof, relying on the Riemann-Hilbert correspondence, is given by Esnault in [**Esn05**].

2.1.12 (Perverse sheaves). — We now introduce an abelian subcategory **Perv**$(X)$ of $D^b_c(X)$ consisting of objects which satisfy a condition on the dimension of the support of their homology sheaves, as well as those of their duals.

Definition 2.1.13. — An object $A$ of $D^b_c(X)$ is called a *perverse sheaf* if, for all integers $q$, one has $\dim(\operatorname{supp} \mathcal{H}^{-q}(A)) \leqslant q$ and $\dim(\operatorname{supp} \mathcal{H}^{-q}(\mathbb{D}(A)) \leqslant q$.

Example 2.1.14. — Here is a typical example of a perverse sheaf. Let $X$ be a variety of dimension $d$, and let $\beta\colon U \to X$ be the inclusion of a smooth open subvariety. Then $\beta_!\mathbb{Q}_U[d]$ is a perverse sheaf on $X$.

2.1.15. — Let ${}^pD^{\leqslant 0}_c(X)$ be the full subcategory of $D^b_c(X)$ of objects $A$ with the property that $\dim(\operatorname{supp} \mathcal{H}^{-q}(A)) \leqslant q$ holds for all integers $q$. Similarly, we define ${}^pD^{\geqslant 0}_c(X)$ as the full subcategory consisting of those objects $A$ such that, for all integers $q$, one has $\dim(\operatorname{supp} \mathcal{H}^{-q}(\mathbb{D}(C)) \leqslant q$. The pair

$$({}^pD^{\leqslant 0}_c(X), {}^pD^{\geqslant 0}_c(X))$$

forms a $t$-structure. Perverse sheaves are precisely the objects of the heart ${}^pD^{\leqslant 0}_c(X) \cap {}^pD^{\geqslant 0}_c(X)$ and form thus an abelian category. This allows one to define cohomology functors

$${}^p\mathcal{H}^n\colon D^b_c(X) \longrightarrow \mathrm{Perv}(X).$$

Theorem 2.1.16 (Artin's Vanishing Theorem for perverse sheaves). — *Let $X$ be an affine variety over $k$ and $A$ a perverse sheaf on $X$. Then $H^q(X, A) = 0$ for all $q > 0$, and $H^q_c(X, A) = 0$ for all $q < 0$.*

THEOREM 2.1.17 (Artin). — *Let $f\colon X \to Y$ be an affine morphism. Then $Rf_*$ is t-right exact and $Rf_!$ is t-left exact for the perverse t-structure.*

2.1.18 (Perverse sheaves on the affine line). — Since we will be mainly dealing with perverse sheaves on the affine line, we now specialize to this setting. A perverse sheaf on the complex affine line is a bounded complex $A$ of sheaves of $\mathbb{Q}$-vector spaces on $\mathbb{A}^1(\mathbb{C})$ with constructible homology sheaves $\mathcal{H}^n(A)$ such that the following three conditions hold:

(a) $\mathcal{H}^n(A) = 0$ for $n \notin \{-1, 0\}$,

(b) $\mathcal{H}^{-1}(A)$ has no non-zero global sections with finite support,

(c) $\mathcal{H}^0(A)$ is a skyscraper sheaf.

2.1.19 (Nearby and vanishing cycles). — Let $A$ be an object of the derived category of constructible sheaves on the complex affine line. Let $S$ be the set of singularities of $A$. For every point $z \in \mathbb{C}$, we denote by $\Phi_z(A)$ the complex of *vanishing cycles* of $A$ at $z$. It is a complex of vector spaces given as follows: Let $\alpha\colon \{z\} \to \mathbb{C}$ be the inclusion, let $\beta\colon D_0 \to \mathbb{C}$ be the inclusion of a small punctured disk around $z$, not containing any of the singularities of $A$, and let $e\colon U \to D_0$ be a universal covering. We define the following complexes of vector spaces (sheaves on a point)

$$\Psi_z(A) = \alpha^*\beta_*e_*e^*\beta^*A[-1], \tag{2.1.19.1}$$

$$\Phi_z(A) = \operatorname{cone}\big(\alpha^*A \longrightarrow \alpha^*\beta_*e_*e^*\beta^*A\big)[-1], \tag{2.1.19.2}$$

where the map in (2.1.19.2) is given by adjunction. We call $\Psi_z(A)$ the complex of *nearby cycles* and $\Phi_z(A)$ the complex of *vanishing cycles* of $A$ at $z$. If $z \notin S$, the complex of vanishing cycles is nullhomotopic. Notice that the definition of nearby and vanishing cycles depends on the choice of a universal covering $U \to D_0$. A different choice $U' \to D_0$ yields different functors $\Psi'_z$ and $\Phi'_z$. Any isomorphism of covers $\gamma : U \to U'$ induces isomorphisms $\gamma^* : \Psi'_z \to \Psi_z$ and $\gamma^* : \Phi'_z \to \Phi_z$. In particular, the deck transformation $U \to U$ coming from the action of the standard generator of $\pi_1(D_0)$ induces an automorphism of vector spaces

$$\gamma_z\colon \Psi_z(A) \longrightarrow \Psi_z(A)$$

called the *local monodromy operator*.

The following lemma is a special case of the general fact that, whenever $A$ is a perverse sheaf, the nearby and vanishing cycles are perverse sheaves as well.

LEMMA 2.1.20. — *Let $A$ be a perverse sheaf on $\mathbb{C}$. The complexes $\Psi_z(A)$ and $\Phi_z(A)$ are homologically concentrated in degree* 0.

PROOF. Let $z \in S$. Without loss of generality, we may restrict $A$ to a small disk $D$ around $z$ not containing any other singularity of $A$. This means that the sheaves $\mathcal{H}^n(A)$ on $D$ are constructible with respect to the stratification $\{z\} \subseteq D$. The complex $A$ fits into the exact truncation triangle $\mathcal{H}^{-1}(A)[1] \to A \to \mathcal{H}^0(A)[0]$, and $\Psi_z(A)$ and $\Phi_z(A)$ are triangulated functors, so it is enough to prove the lemma in the case where $A$ is a skyscraper sheaf sitting in degree 0, and in the case where

$A$ is a constructible sheaf with no non-zero sections with finite support sitting in degree $-1$. For a skyscraper sheaf, $\Phi_z(A)$ is zero and $\Psi_z(A)$ is the stalk at $z$ sitting in degree 0. In the case of a constructible sheaf, $\Phi_z(A)$ is the vector space of global sections of the local system $e^*\beta^*A[-1]$ on the universal cover of $D \setminus \{z\}$, viewed as a complex of sheaves on $\{z\}$ concentrated in degree 0. Finally, the kernel of the adjunction map $\alpha^*A[-1] \longrightarrow \alpha^*\beta_*e_*e^*\beta^*A[-1]$ is the vector space of sections of $A[-1]$ supported on $\{z\}$, but this space is zero because $A$ is perverse. Therefore, the adjunction map is injective and its cone $\Psi_z(A)$ is homologically concentrated in degree 0. □

## 2.2. Computing the cohomology of constructible sheaves on the affine line

In this section, we describe a way to compute the cohomology of constructible sheaves on the affine line using cochains. This is reminiscent of the cochain description of group cohomology, and will be helpful for concrete computations, in particular when we want to handle specific cohomology classes. We will come back to this description in the proof of the key Proposition 2.7.5, and in Section 2.6, where cochains are used to compute the additive convolution of certain perverse sheaves.

2.2.1. — We first interpret constructible sheaves on the complex plane $\mathbb{C}$ in terms of group representations. Let $S \subseteq \mathbb{C}$ be a finite set, $X = \mathbb{C} \setminus S$ its complement, and denote by

$$S \xrightarrow{\alpha} \mathbb{C} \xleftarrow{\beta} X$$

the inclusions. A constructible sheaf $F$ on $\mathbb{C}$ with singularities in $S$ is uniquely described by the following data:

(1) A local system $F_X$ on $X$.

(2) A sheaf $F_S$ on the discrete set $S$, and a morphism of sheaves $F_S \to \alpha^*\beta_*F_X$ on $S$.

Fix a base point $x \in X$, set $G = \pi_1(X, x)$ and denote by $V$ the fibre of $F$ at $x$. The local system $F_X$ determines a representation $\rho\colon G \to \mathrm{GL}(V)$ which characterises $F_X$ up to isomorphism. The sheaf $F_S$ is given by a collection of vector spaces $(V_s)_{s\in S}$. For every path $p\colon [0,1] \to \mathbb{C}$ with $p(0) = s$, $p(1) = x$ and $p(t) \in X$ for $t > 0$, the gluing data (2) determines a linear map $\rho_s(p)\colon V_s \to V$ called *cospecialisation*. If now $\alpha$ and $\beta$ denote the inclusions

$$\{0\} \xrightarrow{\alpha} [0,1] \xleftarrow{\beta} (0,1],$$

then $\rho_s(p)$ is the linear map $V_s \to \alpha^*\beta_*(p|_{(0,1]})^*F_X$ composed with the canonical isomorphism

$$\alpha^*\beta_*(p|_{(0,1]})^*F_X \cong \Gamma((\tau|_{(0,1]})^*F_X) \cong V.$$

The linear map $\rho_s(p)$ only depends on the class of $p$ up to homotopies in $X$ leaving $p(0) = s$ and $p(1) = x$ fixed. This makes sense despite the fact that $s$ is not in $X$. Write

$$P_s = \{\text{paths from } s \text{ to } x \text{ in } X\}/_{\simeq_{\text{homotopy}}}.$$

for the set of these classes. The fundamental group $G$ acts transitively on $P_s$ by concatenation of paths, and for $g \in G$ and $p \in P_s$ the relation $\rho_s(gp) = \rho(g)\rho_s(p)$ holds. We may thus describe constructible sheaves on $\mathbb{C}$ with singularities in $S$, once a base point $x$ is chosen, by the following data:

(1') A vector space $V$, and a linear representation $\rho\colon G \to \mathrm{GL}(V)$.

(2') For every $s \in S$, a vector space $V_s$ and, for every path $p \in P_s$, a cospecialisation map $\rho_s(p)\colon V_s \to V$ such that $\rho_s(gp) = \rho(g)\rho_s(p)$ holds for all $p \in P_s$ and all $g \in G$.

Having $S$ and $x$ fixed, the tuples $(V, \rho, (V_s, \rho_s)_{s\in S})$ form an abelian category in the evident way, which is equivalent to the category of constructible sheaves on $\mathbb{C}$ with singularities contained in $S$. We can now forget about the geometric origin of $G$ and the $P_s$, and are led to the following definition.

Definition 2.2.2. — Let $G$ be a group, and let $P_S = (P_s)_{s\in S}$ be a finite, possibly empty collection of non-empty $G$-sets. A *representation* of $(G, P_S)$ consists of a vector space $V$ and vector spaces $(V_s)_{s\in S}$, a group homomorphism $\rho\colon G \to \mathrm{GL}(V)$ and maps $\rho_s : P_s \to \mathrm{Hom}(V_s, V)$, satisfying $\rho_s(gp) = \rho(g)\rho_s(p)$ for all $g \in G$ and $p \in P_s$. Morphisms of representations are defined in the evident way, and we denote the resulting category by

$$\mathbf{Rep}(G, P_S).$$

2.2.3. — The category of representations of $(G, P_S)$ is an abelian category, and it is indeed the category of sheaves on an appropriate site. Given a representation $V$ of $(G, P_S)$, we call *invariants* the subspace

$$V^{(G,P_S)} \subseteq V \oplus \bigoplus_{s\in S} V_s$$

consisting of those tuples $(v, (v_s)_{s\in S})$ satisfying $gv = v$ for all $g \in G$ and $p_s v_s = v$ for all $p_s \in P_s$. Here, as we shall do from now on if no confusion seems possible, we supressed $\rho$ and $\rho_s$ from the notation. Associating with a representation its space of invariants defines a left exact functor from $\mathbf{Rep}(G, P_S)$ to the category of vector spaces. We can thus define cohomology groups

$$H^n(G, P_S, V)$$

using the right derived functor of the invariants functor. As for ordinary group cohomology, there is an explicit, functorial chain complex which computes this cohomology. Define

$$\begin{aligned} C^0(G, P_S, V) &= V \oplus \bigoplus_{s\in S} V_s, \\ C^n(G, P_S, V) &= \mathrm{Maps}(G^n, V) \oplus \bigoplus_{s\in S} \mathrm{Maps}(G^{n-1} \times P_s, V), \quad n \geqslant 1. \end{aligned}$$

and call elements of $C^n(G, P_S, V)$ *cochains*. Alternatively, we will also think of cochains as functions from the disjoint union of $G^n$ and the $G^{n-1} \times P_s$ to $V$. This can make notations shorter. Define differentials

$$C^0(G, P_S, V) \xrightarrow{d^0} C^1(G, P_S, V) \xrightarrow{d^1} C^2(G, P_S, V) \xrightarrow{d^2} \cdots$$

as follows. We set

$$d^0(v,(v_s)_{s\in S})(g) = v - gv \qquad \text{and} \qquad d^0(v,(v_s)_{s\in S})(p_s) = v - p_s v_s$$

and, for $n > 0$ and $c \in C^n(G, X_S, V)$, we define $d^n c$ by the usual formula

$$\begin{aligned}(d^n c)(g_1,\ldots,g_n,y) &= g_1 c(g_2,\ldots,g_n,y) + \\ &+ \sum_{i=1}^{n-1}(-1)^i c(g_1,\ldots,g_i g_{i+1},\ldots,g_n,y) + (-1)^n c(g_1,\ldots,g_n y) + (-1)^{n+1} c(g_1,\ldots,g_n),\end{aligned}$$

where $y$ is either an element of $G$ or an element of $P_s$ for some $s \in S$. The verification that the spaces $C^n(G, P_S, V)$ and the differentials $d^n$ form a complex is straightforward. The chain complex $C^*(G, P_S, V)$ depends functorially on the representation $V$ in the evident way. The kernel of $d^0$ is the space of invariants, and if $S$ is empty, we get back the standard cochain complex. For each $n \geqslant 0$, the functor $V \longmapsto C^n(G, P_S, V)$ is exact, and therefore $C^*(G, P_S, V)$ computes the cohomology $H^n(G, P_S, V)$ as intended.

2.2.4. — We keep the notation from paragraph 2.2.3, and have a closer look at the first cohomology group $H^1(G, P_S, V)$. The space of cocycles $Z^1(G, P_S, V) = \ker(d^1)$ is the space of tuples $(c,(c_s)_{s\in S})$ consisting of maps $c : G \to V$ and $c_s : P_s \to V$ satisfying the cocycle relations

$$c(gh) = c(g) + gc(h) \qquad \text{and} \qquad c_s(gp_s) = c(g) + gc_s(p_s)$$

for all $g, h \in G$ and $p_s \in P_s$, and the space of coboundaries $B^1(G, P_S, V) = \mathrm{im}(d^0)$ is the space of those tuples of the form

$$c(g) = v - gv \qquad \text{and} \qquad c_s(p_s) = v - p_s v_s$$

for some $v \in V$ and $v_s \in V_s$. For general $(G, P_S)$ and $V$ nothing more can be said. A particular case is interesting to us: Pick elements $p_s^0 \in P_s$, and suppose that $G$ acts transitively on the sets $P_s$, and that the stabilisers $G_s = \{g \in G \mid gp_s^0 = p_s^0\}$ generate $G$. In that case, the whole cocycle $c$ is determined by the values $c_s(p_s^0)$. Indeed, if $c$ is a cocycle satisfying $c(p_s^0) = 0$ for all $s$, then we have

$$c(gp_s^0) = c(g) + gc(p_s^0) = c(g)$$

for all $g \in G$. In particular we find $c(g) = 0$ for all $g \in G_s$. Since $c : G \to V$ is an ordinary cocycle and the stabilisers $G_s$ generate $G$, we find $c(g) = 0$ for all $g \in G$. But then, since $G$ acts transitively on $P_s$, we find $c(p_s) = 0$ for all $p_s \in P_s$ as well, so $c = 0$. A particular case of this is the situation where $G$ is the free group on generators $\{g_s \mid s \in S\}$, and $P_s = G/\langle g_s\rangle$ is the quotient of $G$ by the equivalence relation $gg_s \sim g$, and $p_s^0$ is the class of the unit element. In that case, the injective map

$$Z^1(G, P_S, V) \to \bigoplus_{s\in S} V \tag{2.2.4.1}$$

sending $c$ to $c(p_s^0)_{s\in S}$ is also surjective, and the complex $C^0(G,P_S,V)\to Z^1(G,P_S,V)$ takes the following shape:

$$\begin{aligned} V\oplus\bigoplus_{s\in S}V_s &\xrightarrow{d}\bigoplus_{s\in S}V \\ v,(v_s)_{s\in S} &\longmapsto (v-p_s^0v_s)_{s\in S}. \end{aligned} \tag{2.2.4.2}$$

This is of course precisely the situation at which we arrived in 2.2.1, where $G$ was the fundamental group of $X=\mathbb{C}\setminus S$ based at $x\in X$, and $P_s$ the $G$-set of homotopy classes of paths from $s\in S$ to $X$. The complex (2.2.4.2) computes thus the cohomology $H^*(\mathbb{A}^1,F)$, where $F$ is the constructible sheaf corresponding to the representation $V$.

2.2.5. — We are particularly interested in constructible sheaves $F$ on $\mathbb{C}$ satisfying $H^*(\mathbb{A},F)=0$, that is, $R\pi_*F=0$ for the map $\pi$ from $\mathbb{C}$ to a point. Let again $S\subseteq\mathbb{C}$ be a finite set containing the singularities of $F$, and regard $F$ as a representation $V$ of $(G,P_S)$ as in 2.2.1. The cohomology $H^n(\mathbb{A}^1,F)\cong H^n(G,P_S,V)$ is zero for $n>1$. Therefore, $R\pi_*F=0$ holds if and only if the differential $d\colon C^0(G,P_S,V)\to Z^1(G,P_S,V)$ is an isomorphism. Explicitly, this means that for all $s\in S$ the map $p_s^0\colon V_s\to V$ is injective for one (hence every) $p_s^0\in P_s$, and that

$$\bigcap_{s\in S}p_s^0V_s=\{0\}\qquad\text{and}\qquad\sum_{s\in S}\dim(V/p_s^0V_s)=\dim(V)$$

holds for one (hence for every) choice of elements $p_s^0\in P_s$. It follows from this description that, given constructible sheaves $F_1$ and $F_2$ on $\mathbb{C}$ such that $R\pi_*F_1=R\pi_*F_2=0$, a morphism $\varphi\colon F_1\to F_2$ which induces an isomorphism between the fibres over $x$ is an isomorphism. More generally, the functor

$$\left\{\begin{matrix}\text{Constructible sheaves } F\\ \text{on } \mathbb{C} \text{ with singularities}\\ \text{in } S \text{ and } R\pi_*F=0\end{matrix}\right\}\to\mathbf{Vec}_{\mathbb{Q}} \tag{2.2.5.1}$$

sending $F$ to its fibre $V=F_x$ is exact and faithful.

Lemma 2.2.6. — *Let $F$ and $G$ be constructible sheaves on $\mathbb{C}$. Suppose that $F$ has no non-zero global sections, and that $G$ has no non-zero global sections with finite support. Then $F\otimes G$ has no non-zero global sections.*

Proof. Choose a sufficiently large finite set $S\subseteq\mathbb{C}$ containing the singularities of both $F$ and $G$. In the notation of 2.2.1, the sheaves $F$ and $G$ correspond to representations $V$ and $W$ of $(G,P_S)$. Fix elements $p_s^0\in P_s$, that is, paths from $s\in S$ to the base point $x$ avoiding $S$ along the way. We get complexes

$$V\oplus\bigoplus_{s\in S}V_s\xrightarrow{d_V}\bigoplus_{s\in S}V\qquad\text{and}\qquad W\oplus\bigoplus_{s\in S}W_s\xrightarrow{d_W}\bigoplus_{s\in S}W$$

with $d_V(v,(v_s)_{s\in S})=(v-p_s^0v_s)_{s\in S}$ and $d_W(w,(w_s)_{s\in S})=(w-p_s^0w_s)_{s\in S}$. The representation of $(G,P_S)$ given by the vector spaces $V\otimes W$ and $(V_s\otimes W_s)_{s\in S}S$ with the diagonal actions

$$g(v\otimes w)=gv\otimes gw\qquad\text{and}\qquad p_s(v_s\otimes w_s)=p_sv_s\otimes p_sw_s.$$

corresponds to the sheaf $F \otimes G$. That $F$ and $G$ have no non-zero sections with finite support means that the maps $p_s^0 \colon V_s \to V$ and $p_s^0 \colon W_s \to W$ are injective, and that $F$ has no non-zero sections means that moreover the intersection of the $p_s^0 V_s$ in $V$ is zero. It follows that $p_s \colon V_s \otimes W_s \to V \otimes W$ is injective for every $s \in S$, hence $F \otimes G$ has no non-zero sections with finite support. We also have

$$\bigcap_{s\in S} p_s^0(V_s \otimes W_s) \subseteq \bigcap_{s\in S}(p_s^0 V_s \otimes W) = \Big(\bigcap_{s\in S} p_s^0 V_s\Big) \otimes W = \{0\} \otimes W = \{0\},$$

hence $F \otimes G$ has no non-zero global sections at all. □

LEMMA 2.2.7. — *Let $F$ and $G$ be constructible sheaves on $\mathbb{C}$ with disjoint sets of singularities. The Euler characteristics of $F$, $G$ and $F \otimes G$ are related by*

$$\chi(F \otimes G) + \mathrm{rk}(F \otimes G) = \mathrm{rk}(G)\chi(F) + \mathrm{rk}(F)\chi(G),$$

*where $\mathrm{rk}(F)$ and $\mathrm{rk}(G)$ are the dimensions of the local systems underlying $F$ and $G$.*

PROOF. Let $S$ and $T$ be the sets of singularities of $F$ and $G$ respectively. Fix a base point $x \in \mathbb{C} \setminus (S \cup T)$ and choose a path from each element of $S \cup T$ to $x$. The cohomology of $F$ and $G$ is then computed by complexes

$$V \oplus \bigoplus_{s\in S} V_s \xrightarrow{d_V} \bigoplus_{s\in S} V \qquad \text{and} \qquad W \oplus \bigoplus_{t\in T} W_s \xrightarrow{d_W} \bigoplus_{t\in T} W$$

and the Euler characteristics of $F$ and $G$ are the Euler characteristics of these complexes. Explicitely, these are

$$\chi(F) = (1 - \#S)n + \sum_{s\in S} n_s \qquad \text{and} \qquad \chi(G) = (1 - \#T)m + \sum_{t\in T} m_t$$

where we set $n = \dim V = \mathrm{rk}(F)$ and $n_s = \dim V_s$, and similarly $m = \dim W = \mathrm{rk}(G)$ and $m_t = \dim W_t$. The constructible sheaf $F \otimes G$ has singularities in $S \cup T$, and its cohomology is computed by the complex

$$(V \otimes W) \oplus \bigoplus_{s\in S}(V_s \otimes W) \oplus \bigoplus_{t\in T}(V \otimes W_t) \xrightarrow{d_{V\otimes W}} \bigoplus_{s\in S}(V \otimes W) \oplus \bigoplus_{t\in T}(V \otimes W),$$

whose Euler characteristic is that of $F \otimes G$. An elementary computation shows the equality

$$\chi(F \otimes G) = (-\#S - \#T + 1)nm + m\sum_{s\in S} n_s + n\sum_{t\in T} m_t = m\chi(F) + n\chi(G) - nm$$

which is what we wanted. □

## 2.3. The category $\mathbf{Perv}_0$

In this section, we introduce the category $\mathbf{Perv}_0$ and derive some of its basic properties. Throughout, we let $\pi\colon \mathbb{A}^1_{\mathbb{C}} \to \mathrm{Spec}(\mathbb{C})$ denote the structure morphism and

$$\mathbf{Perv} = \mathbf{Perv}(\mathbb{A}^1(\mathbb{C}), \mathbb{Q})$$

the abelian category of perverse sheaves with rational coefficients on the complex affine line. Recall that its objects are bounded complexes $C$ of sheaves of $\mathbb{Q}$-vector spaces on $\mathbb{A}^1(\mathbb{C})$ with constructible cohomology and satisfying the conditions from 2.1.18.

Definition 2.3.1. — The category $\mathbf{Perv}_0$ is the full subcategory of $\mathbf{Perv}$ consisting of those objects $C$ with no global cohomology, that is, $R\pi_*C = H^*(\mathbb{A}^1(\mathbb{C}), C) = 0$.

2.3.2. — Here are some premonitions of what is to become of $\mathbf{Perv}_0$. As we shall show in Proposition 2.3.7, the category $\mathbf{Perv}_0$ is abelian, in fact, a thick abelian subcategory of $\mathbf{Perv}$. It will turn out in Proposition 2.4.4 that the inclusion $\mathbf{Perv}_0 \to \mathbf{Perv}$ has a left adjoint $\Pi\colon \mathbf{Perv} \to \mathbf{Perv}_0$. Once we understand the basic structure of objects of $\mathbf{Perv}_0$, we will be able to define a functor *nearby fibre at infinity*

$$\Psi_\infty\colon \mathbf{Perv}_0 \to \mathbf{Vec}_{\mathbb{Q}}$$

which is exact and faithful. As a consequence, $\mathbf{Perv}_0$ is artinian and noetherian, and we can associate a dimension with every object $C$ of $\mathbf{Perv}_0$ by declaring that it is the dimension of the vector space $\Psi_\infty(C)$. In section 2.4 we will introduce a tannakian structure on $\mathbf{Perv}_0$, for which $\Psi_\infty$ is a fibre functor. In section 3.2 we will relate objects of $\mathbf{Perv}_0$ with rapid decay cohomology (1.1.1.2) by establishing a canonical and natural isomorphism

$$H^n_{\mathrm{rd}}(X, f) \;\cong\; \Psi_\infty(\Pi({}^pR^n f_*\underline{\mathbb{Q}}_X)),$$

where $\mathbb{Q}_X$ is the constant sheaf with value $\mathbb{Q}$ on $X$. This isomorphism can be seen as an enrichment of the vector space $H^n_{\mathrm{rd}}(X, f)$ with an additional structure, namely that of an object of $\mathbf{Perv}_0$.

Lemma 2.3.3 ([**KKP08**], proof of Theorem 2.29). — *An object $C$ of the derived category of constructible sheaves on $\mathbb{A}^1(\mathbb{C})$ belongs to $\mathbf{Perv}_0$ if and only if it is of the form $C = F[1]$ for some constructible sheaf $F$ satisfying $R\pi_*F = 0$.*

Proof. If $F$ is a constructible sheaf on $\mathbb{A}^1$, then $\mathcal{H}^n(F[1]) = 0$ for $n \neq -1$ and $\mathcal{H}^{-1}(F[1]) = F$, so to ensure that $F[1]$ is perverse one only needs to check that the condition $R\pi_*F = 0$ implies that $F$ has no non-zero global sections with finite support. This is clear since $F$ has no non-zero global sections at all. Conversely, let $C$ be a perverse sheaf on $\mathbb{A}^1$. Invoking the exact triangle

$$\mathcal{H}^{-1}(C)[1] \to C \to \mathcal{H}^0(C)[0],$$

it suffices to prove that both $\mathcal{H}^0(C)$ and $R\pi_*\mathcal{H}^{-1}(C)$ vanish under the assumption $R\pi_*C = 0$. This will follow from the spectral sequence

$$E_2^{p,q} = H^p(\mathbb{A}^1, \mathcal{H}^q(C)) \Longrightarrow H^{p+q}(\mathbb{A}^1, C).$$

Combining the facts that $\mathcal{H}^n(C) = 0$ for $n \notin \{-1, 0\}$ and $\mathcal{H}^0(C)$ is a skyscraper sheaf with Artin's vanishing theorem 2.1.11, the spectral sequence degenerates at $E_2$ and we have:

$$\begin{aligned} H^{-1}(\mathbb{A}^1, C) &= H^0(\mathbb{A}^1, \mathcal{H}^{-1}(C)), \\ H^0(\mathbb{A}^1, C) &= H^1(\mathbb{A}^1, \mathcal{H}^{-1}(C)) \oplus H^0(\mathbb{A}^1, \mathcal{H}^0(C)). \end{aligned}$$

Therefore, the condition $R\pi_* C = 0$ implies $H^0(\mathbb{A}^1, \mathcal{H}^0(C)) = 0$ and $R\pi_* \mathcal{H}^{-1}(C) = 0$. Since $\mathcal{H}^0(C)$ is a skyscraper sheaf, we necessarily have $\mathcal{H}^0(C) = 0$. □

EXAMPLE 2.3.4. — Let $s \in \mathbb{C}$ be a point, and denote by $j(s)\colon \mathbb{C} \setminus \{s\} \hookrightarrow \mathbb{C}$ the inclusion. The constructible sheaf $j(s)_! j(s)^* \mathbb{Q}$ has trivial cohomology. Hence

$$E(s) = j(s)_! j(s)^* \underline{\mathbb{Q}}[1] \tag{2.3.4.1}$$

defines an object of the category $\mathbf{Perv}_0$. More generally, for every local system $L$ on $\mathbb{C} \setminus \{s\}$, the object $j(s)_! L[1]$ belongs to $\mathbf{Perv}_0$. Conversely, if a constructible sheaf $F$ has trivial cohomology and only one singular fibre, located at the point $s \in \mathbb{C}$, then $F$ is of the form $j(s)_! L$ for the local system $L = j(s)^* F$ on $\mathbb{C} \setminus \{s\}$.

DEFINITION 2.3.5. — We call *nearby fibre at infinity* the functor

$$\begin{aligned} \Psi_\infty \colon \mathbf{Perv}_0 &\longrightarrow \mathbf{Vec}_{\mathbb{Q}} \\ F[1] &\longmapsto \operatorname*{colim}_{r \to +\infty} F(S_r) \end{aligned}$$

defined in the evident way on morphisms. Recall that $S_r$ is the closed half-plane $\{\mathrm{Re}(z) \geqslant r\}$.

REMARK 2.3.6. — Link to nearby cycles

PROPOSITION 2.3.7. — *The category* $\mathbf{Perv}_0$ *is a* $\mathbb{Q}$*-linear abelian category and the functor* nearby fibre at infinity $\Psi_\infty \colon \mathbf{Perv}_0 \to \mathbf{Vec}_{\mathbb{Q}}$ *is faithful and exact.*

PROOF. The category $\mathbf{Perv}_0$ is a full additive subcategory of the abelian $\mathbb{Q}$-linear category of rational perverse sheaves on $\mathbb{A}^1(\mathbb{C})$, so $\mathbf{Perv}_0$ is itself a $\mathbb{Q}$-linear category. If $f\colon F \to G$ is a morphism between constructible sheaves on $\mathbb{C}$ satisfying $R\pi_* F = R\pi_* G = 0$, then one has $R\pi_*(\ker f) = 0$ and $R\pi_*(\operatorname{coker} f) = 0$, as one can read off the long exact sequences associated with the exact triangles

$$[0 \to G] \to [F \to G] \to [F \to 0] \qquad \text{and} \qquad [\ker f \to 0] \to [F \to G] \to [0 \to \operatorname{coker} f],$$

noting that $[0 \to \operatorname{coker} f]$ is quasiisomorphic to $[F/\ker f \to G]$. Thus, kernels and cokernels of a morphism in $\mathbf{Perv}_0$ are its kernel and cokernel in $\mathbf{Perv}$, and if any two objects in an exact sequence in $\mathbf{Perv}$ belong to $\mathbf{Perv}_0$, then so does the third, again because $R\pi_*$ is a triangulated functor.

The functor $\Psi_\infty$ is exact: indeed, pick any exact sequence $0 \to F \to G \to H \to 0$ of constructible sheaves on $\mathbb{A}^1(\mathbb{C})$. For every sufficiently big $r$, the restrictions of these sheaves to $S_r$ are local systems, hence constant sheaves since $S_r$ is simply connected. Thus, for every sufficiently big

$r$, the sequence $0 \to F(S_r) \to G(S_r) \to H(S_r) \to 0$ is exact. Finally, we prove that $\Psi_\infty$ is faithful. Let $f\colon F \to G$ be a morphism of constructible sheaves with vanishing global cohomology such that the induced map $F(S_r) \to G(S_r)$ is the zero map for some $r$. We need to show that $f = 0$, that is, $f_z\colon F_z \to G_z$ for any $z \in \mathbb{C}$. The choice of a path starting at $z$, ending in $S_r$ and avoiding the singularities of $F$ and $G$ induces functorial cospecialisation maps $F_z \to F(S_r)$ and $G_z \to G(S_r)$. By 2.2.5, these maps are injective by the assumption on the vanishing of cohomology, hence $f_z = 0$. □

2.3.8 (Simple objects of $\mathbf{Perv}_0$). — We end this section with a description of the simple objects of the category $\mathbf{Perv}_0$. Later, in [where?] when we have defined tensor products and duals in $\mathbf{Perv}_0$, we will be able to say something about extension groups too.

Lemma 2.3.9. — *Let $F[1]$ be a simple object of $\mathbf{Perv}_0$. Let $S \subseteq \mathbb{C}$ be the set of singular points of $F$ and denote by $j\colon \mathbb{C} \setminus S \to \mathbb{C}$ the inclusion. Then either $S$ consists of a single point and $F = j_!j^*\mathbb{Q}$, or the local system $j^*F$ on $\mathbb{C} \setminus S$ is simple and $F = j_*j^*F$.*

Proof. Suppose first that there exists some $s \in S$ such that $F$ has a non-zero section over $\mathbb{C} \setminus \{s\}$, or in other words, that there exists a non-zero morphism $j(s)^*\mathbb{Q} \to j(s)^*F$. In that case, we obtain a non-zero morphism

$$j(s)_!j(s)^*\mathbb{Q}[1] \to F[1]$$

in the category $\mathbf{Perv}_0$, which must be an isomorphism since $F$ is simple. Let us now suppose that $H^0(\mathbb{C} \setminus \{s\}, j(s)^*F) = 0$ for all $s \in S$. For every $s \in S$, the adjunction morphism $F \to j(s)_*j(s)^*F$ is injective, and in the short exact sequence

$$0 \to F \to j(s)_*j(s)^*F \to G \to 0$$

the sheaf $G$ is a skyscraper sheaf supported at $s$. In the associated long exact sequence

$$0 \to H^0(\mathbb{C}, F) \to H^0(\mathbb{C}, j(s)_*j(s)^*F) \to H^0(\mathbb{C}, G) \to H^1(\mathbb{C}, F) \to \cdots$$

the map $H^0(\mathbb{C}, j^*(s)j(s)_*F) \to H^0(\mathbb{C}, G)$ is an isomorphism and all other terms vanish, because $F[1]$ belongs to $\mathbf{Perv}_0$. Since $j(s)^*F$ has no non-zero sections also $H^0(\mathbb{C}, j(s)_*j(s)^*F)$ and hence $H^0(\mathbb{C}, G)$ is zero, hence $G = 0$ because it is a skyscraper sheaf. It follows that the adjunction morphism $F \to j(s)_*j(s)^*F$ is an isomorphism for all $s \in S$. But then, also the adjunction morphism $F \to j_*j^*F$ is an isomorphism because locally around any $s \in S$ it is. Finally, if $j^*F$ was not simple, say $j^*F = F_1 \oplus F_2$, then we could write $F$ as $j_*F_1 \oplus j_*F_2$. If a direct sum of constructible sheaves has trivial cohomology, then both summands have trivial cohomology, hence $j_*F_1[1]$ and $j_*F_2[1]$ both are objects of $\mathbf{Perv}_0$, which conflicts our hypothesis that $F$ was simple. □

Remark 2.3.10. — For every finite set $S \subseteq \mathbb{C}$ containing at least two elements, there exist local systems on $\mathbb{C} \setminus S$ which do not come from objects in $\mathbf{Perv}_0$. For example, let $L$ be a local system of rank $r > 0$ on $\mathbb{C} \setminus S$ with the property that, for each $s \in S$, the local monodromy operator around $s$, acting on the fibre of $L$ near $s$, has no non-zero fixed points. Then $j_*L = j_!L$ has non-trivial cohomology, in fact $H^1(\mathbb{C}, j_!L)$ is a vector space of dimension $(\#S - 1)r$.

## 2.4. Additive convolution

In this section, we introduce the additive convolution of perverse sheaves on the affine line and prove that $\mathbf{Perv}_0$ is a $\mathbb{Q}$-linear tannakian category with respect to the tensor product given by additive convolution. The nearby fibre at infinity $\Psi_\infty$ will turn out to be a fibre functor in section 2.8.

Definition 2.4.1. — Let $F$ and $G$ be objects of $D^b_c(\mathbb{A}^1)$, the bounded derived category of constructible sheaves on $\mathbb{A}^1$. We define the *additive convolution* of $F$ and $G$ as

$$F * G = R\mathrm{sum}_*(\mathrm{pr}_1^* F \otimes \mathrm{pr}_2^* G)$$

where $\mathrm{sum}\colon \mathbb{A}^2 \to \mathbb{A}^1$ is the summation map, and $\mathrm{pr}_1, \mathrm{pr}_2\colon \mathbb{A}^2 \to \mathbb{A}^1$ are the projection maps. We define the functor $\Pi\colon D^b_c(\mathbb{A}^1) \to D^b_c(\mathbb{A}^1)$ as

$$\Pi(F) = F * j_! j^* \underline{\mathbb{Q}}[1], \tag{2.4.1.1}$$

where $j\colon \mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1$ is the inclusion.

Lemma 2.4.2. — *Let $F$ and $G$ be objects of $D^b_c(\mathbb{A}^1)$. For every $z \in \mathbb{C}$, there is a natural isomorphism*

$$(F * G)_z \xrightarrow{\sim} R\pi_*(F \otimes \tau_z^* G)$$

*in the derived category of vector spaces, where $\tau_z : \mathbb{A}^1 \to \mathbb{A}^1$ is the reflection map $\tau_z(x) = z - x$.*

Proof. Suppose that $F$ and $G$ are constructible sheaves, let $S$ and $T$ be finite sets containing the singular points of $F$ and $G$ respectively, and set $Y = S \times \mathbb{C} \cup \mathbb{C} \times T \to \mathbb{C}$. The following holds:

(1) The summation map $\mathrm{sum}\colon \mathbb{C}^2 \to \mathbb{C}$ is a fibre bundle.

(2) The morphism $\mathrm{sum}\colon Y \to \mathbb{C}$ is proper.

(3) Outside $Y$, the sheaf $\mathrm{pr}_1^* F \otimes \mathrm{pr}_2^* G$ is a local system.

It was observed by Nori [**Nor**, Lemma 2.7] or [**No00**, Proposition 1.3A] that (1), (2) and (3) imply *via* a combination of proper base change and the Künneth formula that the base change morphisms

$$(R\mathrm{sum}_*(\mathrm{pr}_1^* F \otimes \mathrm{pr}_2^* G))_z \to R\pi_*((\mathrm{pr}_1^* F \otimes \mathrm{pr}_2^* G)|_{\mathrm{sum}^{-1}(z)}) \tag{2.4.2.1}$$

are isomorphisms in the derived category of vector spaces. The base change morphism (2.4.2.1) is a natural morphism for arbitrary sheaves or complexes of sheaves $F$ and $G$, hence it follows from a dévissage argument that (2.4.2.1) is an isomorphism also when $F$ and $G$ are objects of the bounded derived category of constructible sheaves. The fibre $\mathrm{sum}^{-1}(z) = \{(x, \tau_z(x)) \mid x \in \mathbb{C}\}$ is an affine line, and with respect to the coordinate $x$ the restriction of $\mathrm{pr}_1^* F \otimes \mathrm{pr}_2^* G$ to this line is the sheaf $F \otimes \tau_z^* G$. Hence we obtain a natural isomorphism as claimed. □

Example 2.4.3. — Let $L$ be a local system on $\mathbb{A}^1 \setminus S$ for some finite set $S$, and consider the perverse sheaf $j_! L[1]$ on $\mathbb{A}^1$, where $j\colon \mathbb{A} \setminus S \to \mathbb{A}^1$ is the inclusion. We have seen that $\Pi(j_! L[1])$ is of the form $F[1]$ for some constructible sheaf $F$, and we want to understand $F$. The fibre of $F$ at a point $z \in \mathbb{C}$ is the cohomology group

$$F_z = H^1(\mathbb{C}, j(z)_! j(z)^* j_! L),$$

where $j(z)$ is the inclusion of $\mathbb{C} \setminus \{z\}$ into $\mathbb{C}$. The sheaf $j(z)_! j(z)^* j_! L$ is given by the local system $L$ outside $S \cup \{z\}$, and with trivial fibres at each point of $S \cup \{z\}$. We see that $F$ is given by a local system of rank $\#S \cdot \mathrm{rank}(L)$ on $\mathbb{C} \setminus S$, and that its fibre at $s \in S$ is a vector space of dimension $(\#S - 1) \cdot \mathrm{rank}(L)$. Later we will see how to effectively calculate the monodromy of the local system given by $F$ on $\mathbb{C} \setminus S$.

Proposition 2.4.4. — *Let $F$ and $G$ be objects of $D^b_c(\mathbb{A}^1)$.*

*(1) There is a natural isomorphism $R\pi_*(F * G) \cong R\pi_*(F) \otimes R\pi_*(G)$ in the derived category of vector spaces.*

*(2) If $F$ is perverse and $G$ is an object of $\mathbf{Perv}_0$, then $F * G$ is an object of $\mathbf{Perv}_0$. In particular, the endofunctor $F \longmapsto F * G$ on the derived category of constructible sheaves is exact for the perverse $t$-structure.*

*(3) The functor $\Pi$ from* (2.4.1.1) *sends* $\mathbf{Perv}$ *to* $\mathbf{Perv}_0$ *and is left adjoint to the inclusion* $\mathbf{Perv}_0 \to \mathbf{Perv}$.

*(4) If $F$ is an object of $\mathbf{Perv}_0$, then the canonical morphism $F \to \Pi(F)$ is an isomorphism.*

Proof. Denote by $\pi^2 \colon \mathbb{C}^2 \to \mathrm{Spec}(\mathbb{C})$ the structure morphism and set $F \boxtimes G = \mathrm{pr}_1^* F \otimes \mathrm{pr}_2^* G$. The composite isomorphism

$$R\pi_*(F * G) \cong R\pi^2_*(\mathrm{pr}_1^* F \otimes \mathrm{pr}_2^* G) \cong R\pi^2_*(\mathrm{pr}_1^* F) \otimes R\pi^2_*(\mathrm{pr}_2^* G) \cong R\pi_*(F) \otimes R\pi_*(G)$$

yields (1). The second isomorphism is explained by the fact that a tensor product of flasque sheaves is flasque, and that for arbitrary sheaves $F$ and $G$ we have $\pi^2_*(F \boxtimes G) \cong \pi_* F \otimes \pi_* G$ (the presheaf tensor product is already a sheaf).

Now suppose that $F$ and $G$ are perverse, and that $R\pi_* G = 0$ holds. By (1) we have $R\pi_*(F * G) = 0$, so it will be enough to convince ourselves that $F * G$ is a constructible sheaf placed in degree $-1$. According to Lemma 2.3.3, $F \boxtimes G$ is a complex of constructible sheaves on $\mathbb{C}^2$ in degrees $-2$ and $-1$, hence $F * G$ is supported in cohomological degrees $-2$, $-1$ and $0$. We must show that $\mathcal{H}^{-2}(F * G) = \mathcal{H}^0(F * G) = 0$. For any $z \in \mathbb{C}$, there is according to Lemma 2.4.2 a canonical isomorphism of rational vector spaces

$$\mathcal{H}^q(F * G)_z \cong R^q\pi_*(F \otimes \tau_z^* G).$$

The complex of sheaves $C = F \otimes \tau_z^* G$ is cohomologically supported in degrees $-2$ and $-1$ and

$$\mathcal{H}^{-2}(C) = \mathcal{H}^{-1}(F) \otimes \mathcal{H}^{-1}(\tau_z^* G), \qquad \mathcal{H}^{-1}(C) = \mathcal{H}^0(F) \otimes \mathcal{H}^{-1}(\tau_z^* G).$$

Using the spectral sequence $R^p\pi_*(\mathbb{A}^1, \mathcal{H}^q(C)) \Rightarrow R^{p+q}\pi_* C$, we compute

$$\mathcal{H}^{-2}(F * G)_z = R^{-2}\pi_* C = H^0(\mathcal{H}^{-1}(F) \otimes \mathcal{H}^{-1}(G)),$$
$$\mathcal{H}^0(F * G)_z = \pi_* C = H^1(\mathcal{H}^0(F) \otimes \mathcal{H}^{-1}(G)).$$

Since $F$ is perverse and $G$ belongs to $\mathbf{Perv}_0$, the sheaf $\mathcal{H}^{-1}(F)$ has no non-zero sections with finite supported and $\mathcal{H}^{-1}(G)$ has no global sections, Lemma 2.2.6 implies that $\mathcal{H}^{-2}(F*G)_z = 0$. Since $F$ is perverse, $\mathcal{H}^0(F)$ is a skyscraper sheaf, hence so is $\mathcal{H}^0(F)\otimes\mathcal{H}^{-1}(G)$, thus $\mathcal{H}^0(F*G)_z = 0$.

We have already seen in Example 2.3.4 that $j_!j^*\underline{\mathbb{Q}}[1]$ has trivial cohomology, so it is an object of $\mathbf{Perv}_0$. We need to find a natural isomorphism

$$\mathrm{Hom}(\Pi(F), G) \cong \mathrm{Hom}(F, G) \tag{2.4.4.1}$$

for all perverse sheaves $F$ and $G$ with $R\pi_*G = 0$. Let $i\colon \{0\}\to\mathbb{C}$ be the inclusion. The canonical exact sequence $0\to j_!j^*\underline{\mathbb{Q}}\to\underline{\mathbb{Q}}\to i_*i^*\underline{\mathbb{Q}}\to 0$ induces for any object $F$ in $D^b_c(\mathbb{A}^1)$ the following exact triangle:

$$R\mathrm{sum}_*(F\boxtimes j_!j^*\underline{\mathbb{Q}})\to R\mathrm{sum}_*(F\boxtimes\underline{\mathbb{Q}})\to R\mathrm{sum}_*(F\boxtimes i_*i^*\underline{\mathbb{Q}})\to R\mathrm{sum}_*(F\boxtimes j_!j^*\underline{\mathbb{Q}})[1].$$

The complex $R\mathrm{sum}_*(F\boxtimes i_*i^*\underline{\mathbb{Q}})$ is just $F$, and $R\mathrm{sum}_*(F\boxtimes\underline{\mathbb{Q}})$ is the complex of constant sheaves $\pi^*R\pi_*F$, so we may rewrite the triangle as follows:

$$\Pi(F)[-1]\to\pi^*R\pi_*F\to F\to\Pi(F). \tag{2.4.4.2}$$

The triangle is functorial in $F$, hence produces a natural map $F\to\Pi(F)$ which is an isomorphism if $R\pi_*F = 0$. The adjunction (2.4.4.1) sends a morphism $\Pi(F)\to G$ to the composite $F\to\Pi(F)\to G$, and in the opposite direction a morphism $F\to G$ to the induced morphism $\Pi(F)\to\Pi(G)$ composed with the isomorphism $G\cong\Pi(G)$. This shows (3) and (4). □

From (2.4.4.2) we immediately derive:

Corollary 2.4.5. — *An object $F$ of $D^b_c(X)$ satisfies $\Pi(F) = 0$ if and only if $F$ is constant.*

2.4.6. — There is a variant of additive convolution defined using direct image with compact support, namely

$$F *_! G = R\mathrm{sum}_!(F\boxtimes G).$$

Verdier duality exchanges the two convolutions, in the sense that

$$\mathbb{D}(F*G) = \mathbb{D}(F) *_! \mathbb{D}(G).$$

The object $F *_! G$ of $D^b_c(\mathbb{A}^1)$ is in general not a perverse sheaf, even if $F$ and $G$ belong to $\mathbf{Perv}_0$. One has however the following, which was already proved in [**KS11**, Lemma 4.1]:

Lemma 2.4.7. — *Let $F$ be a perverse sheaf and $G$ an object in $\mathbf{Perv}_0$. Then the forget supports map $F *_! G\to F*G$ induces an isomorphism $\Pi(F *_! G)\cong F*G$.*

Proof. Let $\lambda\colon\mathbb{A}^2\to\mathbb{A}^1\times\mathbb{P}^1$ be the open immersion sending $(x, y)$ to $(x+y, [1\colon x-y])$ and $\kappa\colon\mathbb{A}^1\to\mathbb{A}^1\times\mathbb{P}^1$ the complementary closed immersion. The composition of $\lambda$ with the projection $p$ to the first coordinate is the summation map $\mathrm{sum}\colon\mathbb{A}^2\to\mathbb{A}^1$ and $\lambda$ is indeed a relative compactification. Therefore, $F *_! G = Rp_*\lambda_!(F \boxtimes G)$. Let $L = R\lambda_*(F \boxtimes G)$ and consider the exact triangle $\lambda_!\lambda^* L \longrightarrow L \longrightarrow \kappa_*\kappa^* L$. Applying $Rp_*$ to it, we find

$$\begin{array}{ccccccc}
Rp_*\kappa_*\kappa^* L[-1] & \to & Rp_*\lambda_!\lambda^* L & \to & Rp_* L & \to & Rp_*\kappa_*\kappa^* L. \\
 & & \| & & \| & & \\
 & & F *_! G & \longrightarrow & F * G & &
\end{array}$$

In view of Corollary 2.4.5, we need to show that $Rp_*\kappa_*\kappa^* L$ is a constant sheaf on $\mathbb{A}^1$. Indeed, $\kappa^* L$ is already a constant sheaf on $\mathbb{A}^1$ because the singularities of $F \boxtimes G$ are horizontal and vertical lines in $\mathbb{A}^2$ which do not meet the line at infinity. $\square$

Lemma 2.4.8. — *For every object $F$ of $D^b_c(\mathbb{A}^1)$, the canonical morphism $F \to \Pi(F)$ induces an isomorphism $\Pi(\mathbb{D}(\Pi(F))) \to \Pi(\mathbb{D}(F))$.*

Proof. The perverse sheaf $\mathbb{D}(E(0)) = R\operatorname{Hom}(E(0), \mathbb{Q}[2])$ is an extension of the skyscraper sheaf $\delta_0$ with fibre $\mathbb{Q}$ at $0$ by the constant sheaf $\mathbb{Q}$ on $\mathbb{A}^1$, hence $\Pi(\mathbb{D}(E(0))) = \Pi(\delta_0) = E(0)$. Using this and Lemma 2.4.7, we obtain a string of natural isomorphisms

$$\begin{aligned}
\Pi(\mathbb{D}(\Pi(F))) &= \Pi(\mathbb{D}(F * E(0))) = \Pi(\mathbb{D}(F) *_! \mathbb{D}(E(0)) = \Pi(\mathbb{D}(F) * \mathbb{D}(E(0)) \\
&= \mathbb{D}(F) * \mathbb{D}(E(0)) * E(0) = \mathbb{D}(F) * \Pi(\mathbb{D}(E(0))) = \mathbb{D}(F) * E(0) = \Pi(\mathbb{D}(F)).
\end{aligned}$$

whose composite is indeed the morphism obtained by applying $\Pi \circ \mathbb{D}$ to $F \to \Pi(F)$. $\square$

Theorem 2.4.9. — *Additive convolution defines a tensor product on the $\mathbb{Q}$-linear abelian category $\mathbf{Perv}_0$ with respect to which $\mathbf{Perv}_0$ is a tannakian category.*

2.4.10. — We have already shown in Proposition 2.3.7 that $\mathbf{Perv}_0$ is a $\mathbb{Q}$-linear abelian category and that $\Psi_\infty$ is faithful and exact. Moreover, by Proposition 2.4.4, the category $\mathbf{Perv}_0$ is stable under additive convolution. The functor

$$*\colon \mathbf{Perv}_0 \times \mathbf{Perv}_0 \to \mathbf{Perv}_0$$

is additive in both variables. It is even exact in both variables: Given an exact sequence $0 \to F \to F' \to F'' \to 0$ and an object $G$ in $\mathbf{Perv}_0$, we get an exact sequence

$$0 \to F \boxtimes G \to F' \boxtimes G \to F'' \boxtimes G \to 0$$

of sheaves on $\mathbb{C}^2$. Applying $R\mathrm{sum}_*$ yields a long exact sequence of constructible sheaves on $\mathbb{C}$, of which only the part

$$0 \to F * G \to F' * G \to F'' * G \to 0$$

is non-zero. The associativity constraint (we choose it) is given by the equality of sheaves on $\mathbb{C}^3$

$$(F \boxtimes G) \boxtimes H = F \boxtimes (G \boxtimes H),$$

and associativity of the sum of complex numbers. The commutativity constraint is given by the equality of sheaves on $\mathbb{C}^2$

$$F \boxtimes G = \sigma_*(G \boxtimes F), \qquad \sigma(x, y) = (y, x)$$

together with $\mathrm{sum} \circ \sigma = \mathrm{sum}$. Therefore, additive convolution defines a tensor product on $\mathbf{Perv}_0$. A neutral object for the tensor product is the object $E(0)$, with unit constraint $E(0) * F = \Pi(F) \cong F$ already given in Proposition 2.4.4.

What remains to be shown is the existence of duals and the compatibility of the nearby fibre at infinity with tensor products. Recall that a *dual* of an object $M$ of a tensor category is an object $M^\vee$, together with a coevaluation morphism $c\colon \mathbb{1} \to M \otimes M^\vee$ such that the composition

$$\mathrm{Hom}(X \otimes M, Y) \longrightarrow \mathrm{Hom}(X \otimes M \otimes M^\vee, Y \otimes M^\vee) \longrightarrow \mathrm{Hom}(X, Y \otimes M^\vee)$$

is bijective. If each object admits a dual, we say that the symmetric monoidal category is closed.

Proposition 2.4.11. — *Let $[-1]\colon \mathbb{A}^1 \to \mathbb{A}^1$ be the involution sending $x$ to $-x$. Given an object $C$ of $\mathbf{Perv}_0$, define*

$$C^\vee = \Pi(\mathbb{D}([-1]^*C)) = \mathbb{D}([-1]^*C) * E(0). \tag{2.4.11.1}$$

*There is a canonical bijection $\mathrm{Hom}(X * C, Y) \cong \mathrm{Hom}(X, Y * C^\vee)$, natural in $X, Y$ and $C$. In particular, $C^\vee$ is a dual of $C$.*

Proof. The statement follows from the conjunction of

(a) $\mathrm{Hom}(E(0), X^\vee * Y) = \mathrm{Hom}(X, Y)$,

(b) $(X * Y)^\vee = X^\vee * Y^\vee$.

Indeed, taking these properties for granted, one has:

$$\mathrm{Hom}(X * C, Y) = \mathrm{Hom}(E(0), (X * C)^\vee * Y) = \mathrm{Hom}(E(0), X^\vee * C^\vee * Y) = \mathrm{Hom}(X, Y * C^\vee).$$

To prove (a), recall that the inclusion of $\mathbf{Perv}_0$ into $\mathbf{Perv}$ has a right adjoint functor $\Pi = - * E(0)$, and notice that $E(0) = \Pi(\delta_0)$ for $\delta_0$ the skyscraper sheaf with fibre $\mathbb{Q}$ at 0. Let $\iota\colon \{x+y = 0\} \hookrightarrow \mathbb{A}^2$ denote the inclusion of the antidiagonal. Then:

$$\begin{aligned}
\mathrm{Hom}(E(0), M^\vee * N) &= \mathrm{Hom}(\delta_0, M^\vee * N) && \text{(adjunction)}\\
&= \mathrm{Hom}(\delta_0, \Pi(\mathbb{D}([-1]^*M)) * N) && \text{(definition of } M^\vee)\\
&= \mathrm{Hom}(\delta_0, \mathbb{D}([-1]^*M) * N) && \text{(Prop. 2.4.4 (2))}\\
&= \mathrm{Hom}(\delta_0, R\mathrm{sum}_*(\mathbb{D}([-1]^*M) \boxtimes N)) && \text{(definition of } *)\\
&= \mathrm{Hom}(\mathrm{sum}^*\delta_0, \mathbb{D}([-1]^*M \boxtimes N) && \text{(adjunction)}\\
&= \mathrm{Hom}(\iota_!\mathbb{Q}, \mathbb{D}([-1]^*M \boxtimes N) && \text{(inspection)}\\
&= \mathrm{Hom}(\mathbb{Q}, \iota^!(\mathbb{D}([-1]^*M \boxtimes N)) && \text{(adjunction)}\\
&= \mathrm{Hom}(\mathbb{Q}, \Delta^!(\mathbb{D}(M) \boxtimes N)) && (\Delta = \iota \circ ([-1], \mathrm{id})).
\end{aligned}$$

To conclude, we use: Let $\Delta\colon X \hookrightarrow X \times X$ be the diagonal embedding. Then, for each pair of objects $F$ and $G$ of the derived category of constructible sheaves, the following holds:

$$\mathrm{Hom}_{D^b_c(X)}(F, G) = \mathrm{Hom}_{D^b_c(X)}(\mathbb{Q}, \Delta^!(\mathbb{D}(F) \boxtimes G)).$$

Using the basic properties of $\mathbb{D}$, we find:

$$\begin{aligned}\operatorname{Hom}(F,G) &= \operatorname{Hom}(F,\mathbb{D}(\mathbb{D}(G))) = \operatorname{Hom}(F, R\operatorname{Hom}(\mathbb{D}(G),\omega_X) = \operatorname{Hom}(F\otimes\mathbb{D}(G),\omega_X)\\ &= \operatorname{Hom}(\mathbb{Q}, R\operatorname{Hom}(F\otimes\mathbb{D}(G),\omega_X)) = \operatorname{Hom}(\mathbb{Q},\mathbb{D}(F\otimes\mathbb{D}(G))).\end{aligned}$$

Therefore, we are reduced to show that $\Delta^!(\mathbb{D}(F)\boxtimes G) = \mathbb{D}(F\otimes\mathbb{D}(G))$, which follows from the relation $A\otimes B = \Delta^*(A\boxtimes B)$ and Verdier duality.

We now turn to property (b).

$$\begin{aligned}(X*Y)^\vee &= \Pi(\mathbb{D}[-1]^*R\mathrm{sum}_*(X\boxtimes Y))\\ &= \Pi(\mathbb{D}R\mathrm{sum}_*([-1]^*X\boxtimes[-1]^*Y))\\ &= \Pi(R\mathrm{sum}_!(\mathbb{D}([-1]^*X)\boxtimes\mathbb{D}([-1]^*Y))) && \text{(Verdier duality)}\\ &= \Pi(X^\vee *_! Y^\vee)\\ &= X^\vee * Y^\vee && \text{(Lemma 2.4.7)}\end{aligned}$$

We are done. □

## 2.5. A braid group action

In the next section, we shall describe the monodromy representation underlying a convolution $F*G$ in terms of the monodromy representations underlying $F$ and $G$. This description involves the action of a braid group on fundamental groups, which is what we aim to describe in the present section.

2.5.1. — Let $S$ and $T$ be finite, not necessarily disjoint sets of points in the complex plane $\mathbb{C}$, and define $S+T=\{s+t\,|\,s\in S, t\in T\}$. A point $u\in\mathbb{C}$ does not belong to $S+T$ is and only if the sets $S$ and $u-T=\{u-t\,|\,t\in T\}$ are disjoint. We write $\overline{\mathbb{C}}$ for the compactification of $\mathbb{C}$ by a circle at infinity, so $\overline{\mathbb{C}}=\mathbb{C}\sqcup S^1$, where a system of open neighbourhoods of $z\in S^1=\{z\in\mathbb{C}\,|\,|z|=1\}$ is given by the sets

$$\{w\in\mathbb{C}\,|\,|w|>R, |\arg(w)-\arg(z)|<\varepsilon\}\sqcup\{z'\in S^1\,|\,|\arg(z')-\arg(z)|<\varepsilon\}$$

for large $R$ and small $\varepsilon$, as illustrated in Figure 2.5.1. The space $\overline{\mathbb{C}}$ is called the real blow-up of $\mathbb{P}^1\mathbb{C}$ at infinity. For a complex number $z$ of norm 1, we will write $z\infty$ for the element of the boundary $S^1$ of $\overline{\mathbb{C}}$ with argument $\arg(z)$.

2.5.2. — For each $u\in\mathbb{C}\setminus(S+T)$ let us denote by $G(u)$ the fundamental group of the space $\overline{\mathbb{C}}\setminus(S\cup(u-T))$ relative to the base point $1\infty$. It is the same as the fundamental group of $\mathbb{C}\setminus(S\cup(u-T))$ with respect to a large real number as base point. The groups $G(u)$ form a local system on $\mathbb{C}\setminus(S+T)$, and we may consider its monodromy. Concretely, pick a base point $u_0\in\mathbb{C}\setminus(S+T)$ and define a group homomorphism

$$\beta:\pi_1(\mathbb{C}\setminus(S+T),u_0)\to\operatorname{Aut}(G(u_0)) \tag{2.5.2.1}$$

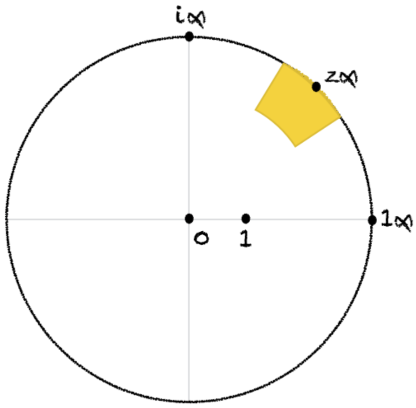



FIGURE 2.5.1. A neighbourhood of $z\infty$

as follows: Given a loop $\gamma : [0,1] \to \mathbb{C} \setminus (S+T)$ based at $u_0$, and a loop $g : [0,1] \to \overline{\mathbb{C}} \setminus (S \cup (u_0 - T))$ based at $1\infty$, we define $\beta(\gamma)(g)$ to be the homotopy class of any loop $g'$ in $\overline{\mathbb{C}} \setminus (S \cup (u_0 - T))$ such that $g' \times 1$ is homotopic to $\tau(g \times 0)\tau^{-1}$ in the space

$$(\overline{\mathbb{C}} \times [0,1]) \setminus \{(z,t) \mid z \in S \cup (\gamma(t) - T)\}$$

where $\tau$ is the path $t \longmapsto (1\infty, t)$. Pick an element $x \in S \cup (u_0 - T)$. We denote by $P_x(u_0)$ the set of homotopy classes of paths from $x$ to $1\infty$ in $\overline{\mathbb{C}} \setminus (S \cup (u_0 - T))$. The group $G(u_0)$ acts on the set $P_x(u_0)$ by composition of paths. Similarly to the action $\beta$ given in (2.5.2.1), there is a canonical action

$$\rho_x : \pi_1(\mathbb{C} \setminus (S+T), u_0) \to \mathrm{Aut}(P_x(u_0)) \tag{2.5.2.2}$$

of $\pi_1(\mathbb{C} \setminus (S+T), u_0)$ on the $G(u_0)$-set $P_x(u_0)$. It is defined by $\beta_x(\gamma)(p) = p'$, where $p'$ is a path from $x$ to $1\infty$ in $\overline{\mathbb{C}} \setminus (S \cup (u_0 - T))$ such that $p' \times 1$ is homotopic to $\tau^{-1}(p \times 0)\tau_x$, where $\tau_x$ is the path $t \longmapsto (x,t)$ if $x \in S$, and $t \longmapsto (\gamma(t) - u_0 - x, t)$ if $x \in u_0 - T$. Figure 2.5.2 illustrates this for $x \in u_0 - T$.

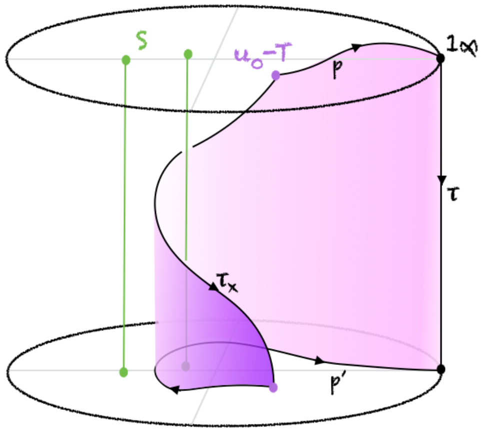



FIGURE 2.5.2. The homotopy $(p' \times 1) \simeq \tau^{-1}(p \times 0)\tau_x$

The two actions $\beta$ and $\beta_x$ are compatible in the sense that the equality

$$\beta_x(\gamma)(gp) = \beta(\gamma)(g)\beta_x(\gamma)(p)$$

holds for all $\gamma \in \pi_1(\mathbb{C} \setminus (S+T), u_0)$, all $g \in G(u_0)$ and all $p \in P_x(u_0)$. Since $G(u_0)$ acts transitively on the set $P_x(u_0)$ the map $\beta_x(\gamma)$ is described by its value on a single element.

Definition 2.5.3. — Let $S$ and $T$ be finite sets of complex numbers and choose $u_0 \in \mathbb{C} \setminus (S+T)$. Define $G(u_0)$ to be the fundamental group of $\overline{\mathbb{C}} \setminus (S \cup (u_0 - T)), 1\infty)$ based at $1\infty$, and for $x \in S \cup (u_0 - T)$ let $P_x(u_0)$ be the set of homotopy classes of paths from $x$ to $1\infty$ in $\overline{\mathbb{C}} \setminus (S \cup (u_0 - T))$. We call *braid-actions* the actions

$$\rho\colon \pi_1(\mathbb{C} \setminus (S+T), u_0) \to \operatorname{Aut}(G(u_0)) \qquad \text{and} \qquad \rho_x : \pi_1(\mathbb{C} \setminus (S+T), u_0) \to \operatorname{Aut}(P_x(u_0))$$

defined in (2.5.2.1) and (2.5.2.2) respectively.

2.5.4. — We can give a hands-on and slightly more practical description of the action (2.5.2.1) by regarding braids as isotopies. Start with a loop $g$ based at $1\infty$ in $\overline{\mathbb{C}}$ avoiding the points $S \cup (u_0 - T)$. Then, as $t$ moves from 0 to 1, the set $\gamma(t) - T$ moves and never touches $S$, and we can deform the ambient space along with this motion, leaving the circle at infinity fixed. In particular the loop $g$ deforms, at all times avoiding points in $S \cup (\gamma(t) - T)$. As $t$ reaches 1, we obtain a new loop in $\overline{\mathbb{C}}$ avoiding the points $S \cup (u_0 - T)$, which we declare to be $\beta(\gamma)(g)$. As a concrete example, take $S = \{0, 1\}$, $T = \{0, i\}$ and $u_0 = 2$, so $S + T = \{0, 1, i, 1 + i\}$ and $S \cup (u_0 - T)$ consists of the four elements $\{0, 1, 2, 2 - i\}$. Now pick a loop $\gamma$ based at $u_0$ avoiding $S + T$ and for each $x \in S \cup (u_0 - T)$ a simple loop $g_x$ around $x$ based at $1\infty$ avoiding $S \cup (u_0 - T)$, as in Figure 2.5.3.

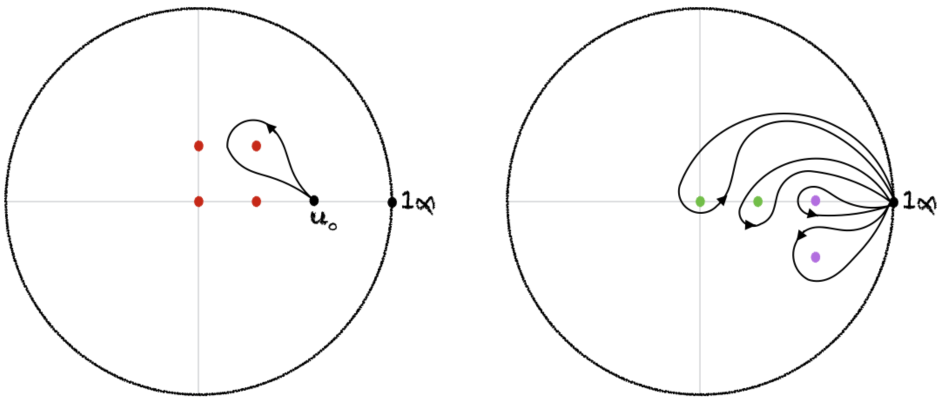



Figure 2.5.3. The loops $\gamma$ (left) and $g_0, g_1, g_2, g_{2-i}$ (right)

The fundamental group $G(u_0)$ is the free group generated by the $g_x$. On the right hand picture, we now move the elements $\{2, 2 - i\}$ of $u_0 - T$ along $\gamma(t) - T$, and deform the loops $g_x$ accordingly. In the left part Figure 2.5.4 we have indicated $g_{2-i}$ and the paths $\gamma(t) - T$ in grey. Deforming $g_{2-i}$ results in a new path $\rho(\gamma)(g_{2-i})$, which is drawn on the right in Figure 2.5.4. This path is $\rho(\gamma)(g_{2-i}) = g_2 g_1^{-1} g_2^{-1} g_{2-i} g_2 g_1 g_2^{-1}$.

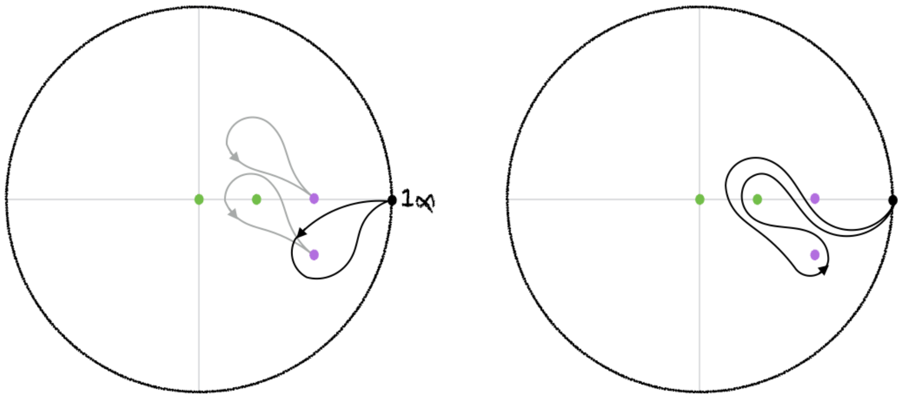



FIGURE 2.5.4. The loop $g_{2-i}$ (left) and $\rho(\gamma)(gv_{2-i})$ (right)

We determine $\rho(\gamma)(g_x)$ for all $x \in \{0, 1, 2, 2-i\}$ in a similar way, and find $\rho(\gamma)$ to be the following automorphism of $G(u_0)$.

$$\begin{array}{rcl} g_0 & \longmapsto & g_0 \\ g_1 & \longmapsto & g_1^{-1} g_2^{-1} g_{2-i}^{-1} g_2 g_1 g_2^{-1} g_{2-i} g_2 g_1 \\ g_2 & \longmapsto & g_2 \\ g_{2-i} & \longmapsto & g_2 g_1^{-1} g_2^{-1} g_{2-i} g_2 g_1 g_2^{-1} \end{array} \tag{2.5.4.1}$$

Notice that for each $x \in \{0, 1, 2, 2-i\}$ we have $\rho(\gamma)(g_x) = h_x \alpha_x h_x^{-1}$ for some $h_x \in G(u_0)$. The description works for the braid action on paths. Let us choose $p_x \in P_x(u_0)$ for $x \in \{0, 1, 2, 2-i\}$ as indicated on the left hand part of Figure 2.5.5. The paths $\rho_x(\gamma)(p_x)$ are the ones on the right.

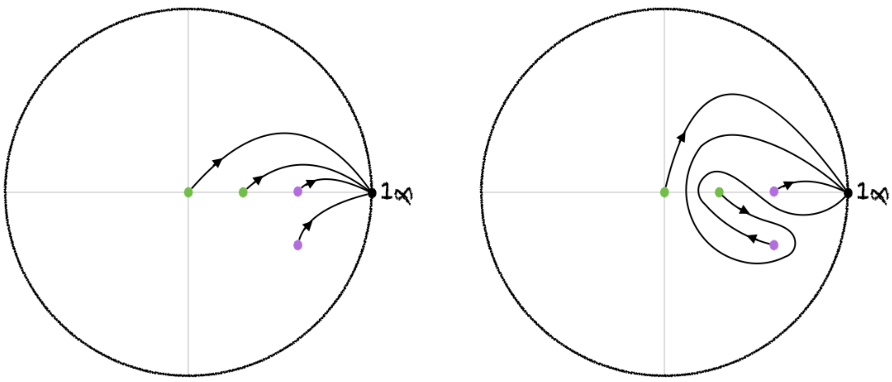



FIGURE 2.5.5. The paths $p_0, p_1, p_2, p_{2-i}$ (left) and their image under $\rho_x(\gamma)$ (right)

The action of $\gamma$ on path spaces $P_x(u_0)$ is uniquely determined by (2.5.2.2) and the following values.

$$\begin{array}{rcl} p_0 & \longmapsto & p_0 \\ p_1 & \longmapsto & g_1^{-1} g_2^{-1} g_{2-i}^{-1} g_2 p_1 \\ p_2 & \longmapsto & p_2 \\ p_{2-i} & \longmapsto & g_2 g_1^{-1} g_2^{-1} p_{2-i} \end{array} \tag{2.5.4.2}$$

Notice that we have indeed $\rho_x(p_x) = h_x p_x$ for the same $h_x$ we found in (2.5.4.1), satisfying $\rho(\gamma)(g_x) = h_x g_x h_x^{-1}$. This is so because of the implicit compatibility between the choices of the loops $g_x$ and that of the paths $p_x$.

2.5.5. — We devote the rest of this section to a practical description of the braid actions, which will be helpful to speed up explicit computations. The idea is to produce in a more or less systematic way the group elements $h_x$ which appeared in (2.5.4.1) and (2.5.4.2). This in turn has to do much with the choice of generators $g_x$ of the fundamental group $G(u_0)$, and on the choice of paths $p_x$.

Given a finite, ordered set of points $X = \{x_1, x_2, \ldots, x_n\}$ in $\mathbb{C}$, we may construct a set of generators of $\pi_1(\overline{\mathbb{C}} \setminus X, 1\infty)$ as follows. Choose distinct points points $u_1, u_2, \ldots, u_n$ on the boundary of $\mathbb{C}$ with increasing arguments $0 < \arg(u_1) < \arg(u_2) < \cdots < \arg(u_n) < 2\pi$, and choose mutually disjoint simple paths[1] $\sigma_m$ from $u_m$ to $x_m$ for each $m$. We refer to such paths as *cuts*. Define $\alpha_m \in \pi_1(\overline{\mathbb{C}} \setminus X, 1\infty)$ to be the class of a simple loop which is disjoint from the cuts $\sigma_m$, except for one transversal positive intersection with $\sigma_m$, and define $\tau_m \in P(x_m)$ to be the class of a simple path from $x_m$ to $1\infty$ which does not intersect any of the cuts. Figure 2.5.6 illustrates this.

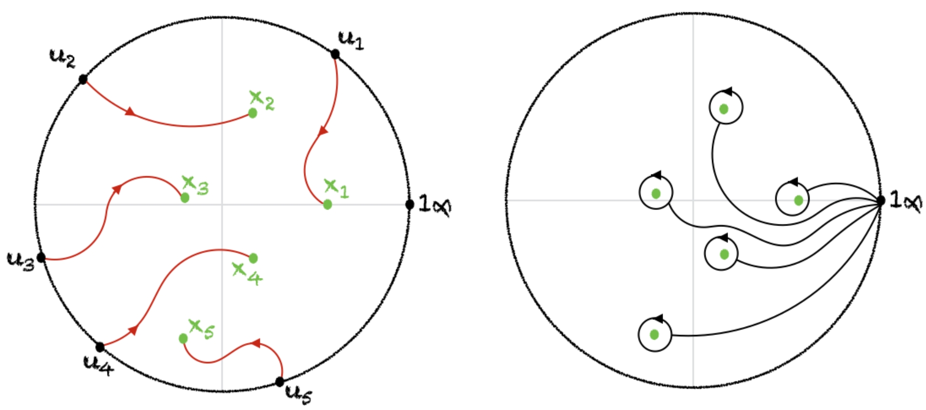



FIGURE 2.5.6. A choice of cuts (left) and the corresponding loops $\alpha_m$ (right)

This cutting up of $\overline{\mathbb{C}} \setminus X$ has the following practical use. Given any loop $\beta$ in $\overline{\mathbb{C}} \setminus X$ based at $1\infty$, we can express its class in $\pi_1(\overline{\mathbb{C}} \setminus X, 1\infty)$ in terms of $\alpha_1, \ldots, \alpha_n$ simply by listing which cuts it crosses with which sign. A reordering of the set $X$ and a subsequent different choice of cuts affects the set of associated generators as follows. Let $\varepsilon$ be a permutation of $\{1, 2, \ldots, n\}$, set $x'_m := x_{\varepsilon(m)}$ and choose cuts $\sigma'_m$ from $u_m$ to $x'_m$. Associated with the cuts $\sigma'_m$ are generators $\alpha'_m$ of the fundamental group $\pi_1(\overline{\mathbb{C}} \setminus X, 1\infty)$, and paths $\tau'_m$ from $x'_m$ to $1\infty$. We can arrange all these paths in such a way that they intersect transversally and only in finitely many points. The path $\tau'_m$ is a path starting at $x'_m = x_{\varepsilon(m)}$ and ending at $1\infty$. On its way, it crosses the cuts $\sigma_k$ in a certain order, maybe multiple times, each time with a positive or negative orientation. We encode

[1] A *simple* path is a path which is also an immersion, that is, a homeomorphism onto its image.

this crossing sequence in a word with letters $\alpha_1, \ldots, \alpha_n$ as

$$w_m = \cdots \alpha_{k_2}^{e_2} \alpha_{k_1}^{e_1}$$

meaning that $\tau'_m$ crosses on its way from $x'_m$ to $1\infty$ first the cut $\sigma_{k_1}$ with orientation sign $e_1$, then the cut $\sigma_{k_2}$ with sign $e_2$ and so on. The loops $\alpha'_m$ and paths $\tau'_m$ are then given by

$$\begin{aligned} \alpha'_m &= w_m \alpha_{\varepsilon(m)} w_m^{-1} \\ \tau'_m &= w_m \tau_m \end{aligned}$$

in terms of the loops $\alpha_m$ and paths $\tau_m$.

2.5.6. — Let $n \geqslant 2$ be an integer. We denote by $B_n$ the abstract braid group on $n$ strands, that is, the group generated by elements $\beta_1, \beta_2, \ldots, \beta_{n-1}$, subject to the following relations:

$$\beta_i \beta_j = \beta_j \beta_i \qquad \text{for all } 1 \leqslant i \leqslant n-1 \text{ with } |i-j| \geqslant 2 \tag{2.5.6.1}$$

$$\beta_i \beta_{i+1} \beta_i = \beta_{i+1} \beta_i \beta_{i+1} \qquad \text{for all } 1 \leqslant i < n-1 \tag{2.5.6.2}$$

One should think of $\beta_i$ as the braid which twists the strands $i$ and $i+1$, and leaves all other strands as they are. There is a canonical surjective homomorphism $B_n \to \mathfrak{S}_n$, sending $\beta_i$ to the transposition $(i, i+1)$. The kernel of this homomorphism is called the *pure braid grouop* on $n$ strands, and denoted by $P_n$. Let $X = \{x_1, \ldots, x_n\}$ be a finite ordered set of complex numbers. We define an action of $B_n$ on $\pi_1(\overline{\mathbb{C}} \setminus X, 1\infty)$ by choosing cuts of $\overline{\mathbb{C}} \setminus X$ as in 2.5.5, and setting

$$\beta_i(\alpha_m) = \begin{cases} \alpha_i^{-1} \alpha_{i+1} \alpha_i & \text{if } m = i \\ \alpha_i & \text{if } m = i+1 \\ \alpha_m & \text{if } m \notin \{i, i+1\} \end{cases} \tag{2.5.6.3}$$

for the associated generators $\alpha_1, \ldots, \alpha_n$. To see that this gives a well defined action, we have to check compatibility with the relations (2.5.6.1) and (2.5.6.2).

2.5.7. — In 2.5.2 we have introduced the action of $\pi_1(\mathbb{C} \setminus (S+T), u_0)$ on the fundamental group $G(u_0)$ as a monodromy action on a local system of groups. We now want to have a closer look at local monodromy operators. This means that we take for $u_0$ a point close to an element $z_0 \in S+T$, and for $\gamma$ a simple, positively oriented loop around $z_0$.

Let $(s_1, t_1), (s_2, t_2), \ldots, (s_m, t_m)$ be the set of those pairs $(s, t) \in S \times T$ satisfying $s + t = z_0$. For generic $S$ and $T$, there is only one such pair. To describe the action of $\gamma$ on $G(u_0)$, let us order the set $S + (u_0 - T)$ in any way starting as follows.

$$s_1, (u_0 - t_1), s_2, (u_0 - t_2), \ldots, s_m, (u_0 - t_m), \ \ldots \text{other elements of } S + (u_0 - T)$$

From this ordering of $S+(u_0-T)$ we obtain a set of generators of $G(u_0)$ as constructed in 2.5.5, and a path from each element of $S + (u_0 - T)$ to $1\infty$. Let us denote the first $2m$ of these generators by $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_m, \beta_m$, and the paths by $\tau_1, \sigma_1, \tau_2, \sigma_2, \ldots, \tau_m, \sigma_m$. The local monodromy action is given, in terms of these generators and paths, by

$$\begin{aligned} \rho(\gamma)(\alpha_i) &= \alpha_i^{-1}\beta_i^{-1}\alpha_i\beta_i\alpha_i & \rho(\gamma)(\beta_i) &= \alpha_i^{-1}\beta_i\alpha_i \\ \rho_{s_i}(\gamma)(\tau_i) &= \alpha_i^{-1}\beta_i^{-1}\tau_i & \rho_{u_0-t_i}(\gamma)(\sigma_i) &= \alpha_i^{-1}\sigma_i \end{aligned} \tag{2.5.7.1}$$

On all other generators of $G(u_0)$ and paths, the monodromy action is trivial.

2.5.8. — For a base point $u_0$ close to an element of $S+T$, the formulas ?? and (2.5.7.1) allow us to compute algebraically, without falling back to drawings, the local monodromy action of $\pi_1(\mathbb{C}\setminus(S+T),u_0)$ on $G(u_0)$ in terms of generators of $G(u_0)$ associated with any ordering of the set $S+(u_0-T)$. In order to understand the global monodromy action of $\pi_1(\mathbb{C}\setminus(S+T),u_0)$ on $G(u_0)$ for an arbitrary $u_0\notin S+T$, we need to understand parallel transport from $G(u_0)$ to $G(u_1)$ along a simple path from $u_0$ to $u_1$ in $\mathbb{C}\setminus(S+T)$. Having chosen an ordering of the set $S\cup(u_0-T)$, we obtain an ordering of $S\cup(u_1-T)$. Parallel transport along a simple path from $u_0$ to $u_1$ sends each of the generators of $G(u_0)$ to the corresponding one on $G(u_1)$ in the obvious way, and same works for the distinguished elements in the path spaces. Having chosen a set of simple loops generating $\pi_1(\mathbb{C}\setminus(S+T),u_0)$, this allows us to compute the whole action of $\pi_1(\mathbb{C}\setminus(S+T),u_0)$ on $G(u_0)$ in a purely algebraic way. We illustrate this on an example.

## 2.6. Computing fibres and monodromy of a convolution

In this section, we give an explicit description of the additive convolution of perverse sheaves in terms of group representations. We are interested in the particular case of convolution of two perverse sheaves which both belong to **Perv**$_0$, since additive convolution is the tensor product in the tannakian category **Perv**$_0$.

Let $F$ and $G$ be constructible sheaves on the complex affine line with vanishing cohomology, so that $F[1]$ and $G[1]$ are objects of **Perv**$_0$. Let $S$ be the set of singularities of $F$, so that on the complement of $S$ the sheaf $F$ is given by a local system, or alternatively by a representation of the fundamental group of $\mathbb{C}\setminus S$. Let $T$ be the set of singularities of $G$. The convolution $F[1]*G[1]$ is again an object of **Perv**$_0$, hence described by a constructible sheaf with vanishing cohomology. The singularities of $F[1]*G[1]$ are contained in the set $S+T=\{s+t\mid s\in S, t\in T\}$, hence $F[1]*G[1]$ defines a local system on $\mathbb{C}\setminus(S+T)$, or alternatively by a representation of the fundamental group of $\mathbb{C}\setminus(S+T)$. We want to understand the latter representation in terms of those determined by $F$ and $G$.

2.6.1. — Let us fix the following material for this section. Constructible sheaves $A$ and $B$ on $\mathbb{C}$ with vanishing cohomology, with finite sets of singularities $S_A\subseteq\mathbb{C}$ and $S_B\subseteq\mathbb{C}$ respectively. Set

$$G_A := \pi_1(\overline{\mathbb{C}}\setminus S_A, 1\infty) \qquad \text{and} \qquad P_{A,s} := \pi_1(\overline{\mathbb{C}}\setminus S_A, s, 1\infty)$$

so that in terms of group representations, the sheaf $A$ be given by vector spaces $V_A = \Psi_\infty(A[1])$ and $(V_{A,s})_{s\in S_A}$ and

$$\rho_A : G_A \to \mathrm{GL}(V_A), \qquad \rho_{A,s} : P_{A,s} \to \mathrm{Hom}(V_{A,s}, V_A)$$

as described in 2.2.1. Let $B$ be given by similar data. Fix a real $r$ larger than $|s|$ for any $s \in S_A \cup S_B$, so that we have in particular $r \notin S \cup (2r - T)$ and $2r \notin S + T$. Set $C := A \otimes \tau_{2r}^* B$, so the singularities of $C$ are $S_C = S_A \cup (2r - S_B)$. Identify the fundamental group $\pi_1(\overline{\mathbb{C}} \setminus S_C, r)$ with the free product of the fundamental groups $G_A$ and $G_B$ using the Seifert–van-Kampen theorem as we already did in the proof of Proposition 2.7.5 where we showed compatibility of $\Psi_\infty$ with tensor products. Let $\omega$ be the path $t \longmapsto re^{\pi i t}$ from $r$ to $-r$ in $\overline{\mathbb{C}}$, and prolong it along real half-lines $[-\infty, -r]$ and $[r, \infty]$ whenever necessary. We use $\omega$ to identify $G_B = \pi_1(\overline{\mathbb{C}} \setminus S_B, 1\infty)$ with $\pi_1(\overline{\mathbb{C}} \setminus S_B, -1\infty)$ and $V_B = \Psi_\infty(B[1])$ with $V_B^- := \Psi_{-\infty}(B[1])$. Finally use the path $\tau_{2r} \circ \omega$ between $1\infty$ and $r$ to identify $G_C := \pi_1(\overline{\mathbb{C}} \setminus S_C, r)$ with $G_A * G_B = \pi_1(\overline{\mathbb{C}} \setminus S_C, r)$ as illustrated in Figure 2.6.7.

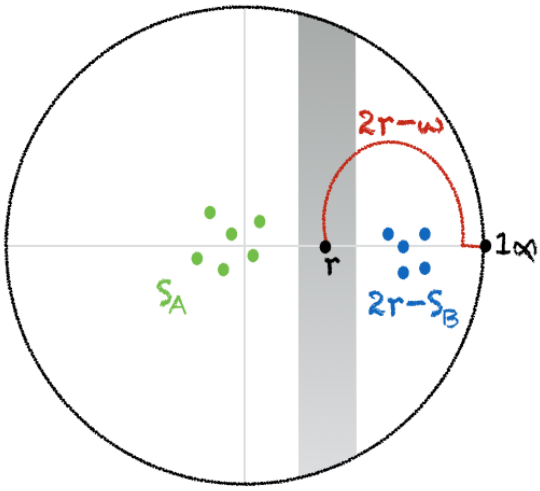



FIGURE 2.6.7. The set $S_C = S_A \cup (2r - S_B)$ and the path $\tau_{2r} \circ \omega$

2.6.2. — Let us describe the constructible sheaf $C = A \otimes \tau_{2r}^* B$ in terms of representations of the group $G_C$ and the path spaces $P_{C,s}$ for $s \in S_C$. For $s \in S_A$, we continue paths from $s$ to $r$ along the path $\tau_{2r} \circ \omega$ to obtain paths from $s$ to $1\infty$. This way, the set $P_{C,s}$ of homotopy classes of paths from $s$ to $1\infty$ in $\overline{\mathbb{C}} \setminus S_C$ is the amalgamed product $G_C *_{G_A} P_{A,s}$. Similarly, any path $p$ from $s \in S_B$ to $r$ yields a path $\tau_{2r} \circ p$ from $2r - s$ to $r$, and we can continue it along $\tau_{2r} \circ \omega$ to obtain a path from $2r - s$ to $1\infty$. This way, $P_{C,2r-s}$ is the amalgamed product $G_C *_{G_B} P_{B,s}$.

The representation data for $C$, using $1\infty \in \overline{\mathbb{C}}$ as the chosen base point, consists of the vector spaces

$$V_C = V_A \otimes V_B^- \cong V_A \otimes V_B \qquad \text{and} \qquad V_{C,s} = \begin{cases} V_{A,s} \otimes V_B & \text{if } s \in S_A \\ V_A \otimes V_{B,2r-s} & \text{if } s \in 2r - S_B \end{cases}$$

and actions $\rho_C$ and $\rho_{C,s}$

$$\rho_C(g) = \begin{cases} \rho_A(g) \otimes \mathrm{id}_{V_B} & \text{if } g \in G_A \\ \mathrm{id}_{V_A} \otimes \rho_B(g) & \text{if } g \in G_B \end{cases} \qquad \text{and} \qquad \rho_{C,s}(p_s) = \begin{cases} \rho_{A,s}(p_s) \otimes \mathrm{id}_{V_B} & \text{if } p_s \in P_{A,s} \\ \mathrm{id}_{V_A} \otimes \rho_B(g) & \text{if } g \in G_B \end{cases}$$

where we use $\omega$ for the identifications $V_B^- \cong V_B$ and $G_C = G_A * G_B$.

2.6.3. — The fibre near infinity $\Psi_\infty(A[1] * B[1])$ is canonically isomorphic to the cohomology group

$$(A * B)[1]_{2r} \cong H^1(\mathbb{C}, A \otimes \tau_{2r}^* B) \cong H^1(G_C, P_{C,S_C}), (V_C, V_{C,S})) \tag{2.6.3.1}$$

The group $\pi_1(\overline{\mathbb{C}} \setminus (S_A + S_B), 1\infty) = \pi_1(\overline{\mathbb{C}} \setminus (S_A + S_B), 2r)$ acts on $(A * B)[1]_{2r}$ by monodromy. It also acts on $G_C$ and the $G_C$-sets $P_{C,s}$ via the braid group action

$$\beta : \pi_1(\overline{\mathbb{C}} \setminus (S_A + S_B), 2r) \to \mathrm{Aut}(G_C) \qquad \text{and} \qquad \beta_s : \pi_1(\overline{\mathbb{C}} \setminus (S_A + S_B), 2r) \to \mathrm{Aut}(P_{C,s})$$

defined in 2.5.3 in the previous section. The main result of this section, Theorem 2.6.4 below, states that these two actions are compatible. Notice that the statement does not involve in any way the identification $G_C = G_A * G_B$ or the isomorphism $V_B \cong V_B^-$ which we obtained from $\omega$. These were only used to describe the representation data for $C$ in terms of the representation data for $A$ and $B$.

Theorem 2.6.4. — *The monodromy action of $\pi_1(\overline{\mathbb{C}} \setminus (S_A + S_B), 2r)$ on the fibre $(A * B)[1]_{2r}$ corresponds, via the isomorphisms* (2.6.3.1)*, to the action*

$$\pi_1(\overline{\mathbb{C}} \setminus (S_A + S_B), 2r) \to \mathrm{GL}(H^1(G_C, P_{C,S_C}), (V_C, V_{C,S_C})))$$

*sending $\gamma \in \pi_1(\overline{\mathbb{C}} \setminus (S_A + S_B), 2r)$ to the linear map defined by $(c, c_{S_C}) \longmapsto (c \circ \beta(\gamma^{-1}), c_{S_C} \circ \beta_{S_C}(\gamma^{-1}))$ on cocycles.*

Proof. We place ourselves in the slightly more general setting at first. Let $(\gamma_s)_{s \in S}$ be a finite collection of paths $\gamma_s \colon [0,1] \to \mathbb{C}$ which are disjoint at all times $t \in [0,1]$. We also denote by $\gamma_s$ the set $\{(\gamma_s(t), t) \mid t \in [0,1]\}$ and set $S_t = \{\gamma_s(0) \mid s \in S\}$. The $\gamma_s$ are the strands of a braid in $\overline{\mathbb{C}} \times [0,1]$ from $S_0$ to $S_1$. Let $F$ be a sheaf on $\overline{\mathbb{C}} \times [0,1]$ a sheaf which is constructible, in the sense that $F$ is locally constant outside the union of the strands $\gamma_s$, and locally constant on each strand. Let us also assume that $F$ is constant, given by a vector space $V$, on $\{1\infty\} \times [0,1]$. For each time $t \in [0,1]$, the restriction of $F$ to $\overline{\mathbb{C}} \times \{t\}$ is a constructible sheaf with singularities contained in $S_t$.

Let $\pi \colon \overline{\mathbb{C}} \times [0,1] \to [0,1]$ be the projection. The sheaves $R^n\pi_*F$ are locally constant on $[0,1]$, with fibres

$$(R^n\pi_*F)_t = H^n(\overline{\mathbb{C}} \times \{t\}, F_{\overline{\mathbb{C}} \times \{t\}})$$

by proper base change. The parallel transport isomorphism $(R^n\pi_*F)_0 \to (R^n\pi_*F)_1$ is the composite of base isomorphisms

$$H^n(\overline{\mathbb{C}} \times \{0\}, F_{\overline{\mathbb{C}} \times \{0\}}) \xleftarrow{\cong} H^n(\overline{\mathbb{C}} \times [0,1], F) \xrightarrow{\cong} H^n(\overline{\mathbb{C}} \times \{1\}, F_{\overline{\mathbb{C}} \times \{1\}}) \tag{2.6.4.1}$$

induced by inclusions $\overline{\mathbb{C}} \times \{t\} \subseteq \overline{\mathbb{C}} \times [0,1]$ for $t = 0, 1$.

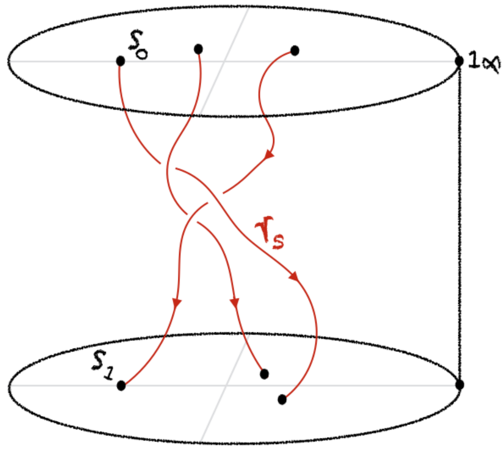



FIGURE 2.6.8. A general braid

Let us contract $\overline{\mathbb{C}} \times [0,1]$ the line $\{1\infty\} \times [0,1]$ to a point, and denote by $G$ be the fundamental group of the complement of $\{\gamma_s \mid s \in S\}$ in the quotient space, with base point $1\infty$. For each $s \in S$ let $P_s$ be the $G$ set of homotopy classes of paths from $\gamma_s$ to $1\infty$, up to homotopies which leave the starting point of the path inside $\gamma_s$ and fix the endpoint $1\infty$. From the sheaf $F$ we obtain a representation of $(G, P_S)$, given by the vector spaces $V = F_{1\infty}$ and $V_s = \Gamma(\gamma_s, F|_{\gamma_1})$, the monodromy representation of $G$ and the cospecialisation maps for each $s \in S$. The cohomology groups $H^n((G, P_S), V)$ are canonically isomorphic to the sheaf cohomology groups $H^n(\overline{\mathbb{C}} \times [0,1], F)$. For each $t \in [0,1]$, let $G_t$ be the fundamental group of $\overline{\mathbb{C}} \setminus S_t$ based at $1\infty$, and denote by $P_{s,t}$ the $G_t$-set of paths in $\overline{\mathbb{C}} \setminus S_t$ from $\gamma_s(t) \in S_t$ to $1\infty$. The inclusion of $\overline{\mathbb{C}} \times \{t\}$ into $\overline{\mathbb{C}} \times [0,1]$ induces an isomorphism $\beta_t\colon (G_t, P_{S,t}) \xrightarrow{\cong} (G, P_S)$. Let $V_t$ denote the representation of $(G_t, P_{S,t})$ corresponding to $F_{\overline{\mathbb{C}} \times \{t\}}$. The specialisation map

$$H^n(\overline{\mathbb{C}} \times [0,1], F) \xrightarrow{\cong} H^n(\overline{\mathbb{C}} \times \{t\}, F_{\overline{\mathbb{C}} \times \{t\}})$$

are, in terms of group cohomology, induced by the morphism of chain complexes

$$\begin{array}{ccccc} C^0((G, P_S), V) & \xrightarrow{d} & C^1((G, P_S), V) & \longrightarrow & \cdots \\ \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi_1} & & \\ C^0((G_t, P_{S,t}), V_t) & \xrightarrow{d} & C^1((G_t, P_{S,t}), V_t) & \longrightarrow & \cdots \end{array}$$

given in degree zero by specialisation

$$\varphi_0 : V \oplus \bigoplus_{s \in V} \Gamma(\gamma_s, F|_{\gamma_s}) \to V \oplus \bigoplus_{s \in V} F_{(\gamma_s(t), t)}$$

and in degrees $n > 0$ by precomposing cocycles with the isomorphism $\beta_t$. In particular, the transport isomorphism (2.6.4.1) is, in terms of cocycles and for $n > 0$, given by sending the class of a cocycle $c \in C^n(G_0, P_{S,0}, V)$ to the class of its composite with the isomorphism

$$(G_1, P_{S,1}) \xrightarrow{\beta_0^{-1}\beta_1} (G_0, P_{S,0}). \qquad (2.6.4.2)$$

Let us now come back to the situation of the theorem. We consider the map $\mu : \overline{\mathbb{C}} \times [0,1] \to \overline{\mathbb{C}}^2$ defined by $\mu(z,t) = (z, \gamma(t) - z)$ for $z \in \mathbb{C}$ and $\mu(z,t) = (z, -z)$ for $z \in \partial\overline{\mathbb{C}}$, and the sheaf $F := \mu^*(A \boxtimes B)$ on $\overline{\mathbb{C}} \times [0,1]$. The fibre of $F$ at $(z,t)$ is $A_z \otimes B_{\gamma(t)-z}$, and in particular the fibre of $F$ at $(1\infty, t)$ is $V_A \otimes V_B^-$ for all $t \in [0,1]$. The sheaf $F$ is constructible in the same sense as before, with respect to the braid with strands $\gamma_s(t) = s$ for $s \in S_A$ and $\gamma_s(t) = \gamma(t) - s$ for $s \in S_B$. The parallel transport isomorphism (2.6.4.1) specialises to the monodromy action, the braid action

$$\beta(\gamma)^{-1} : (G_C, P_{S_C}) \to (G_C, P_{S_C})$$

is the same as the composite isomorphism (2.6.4.2). □

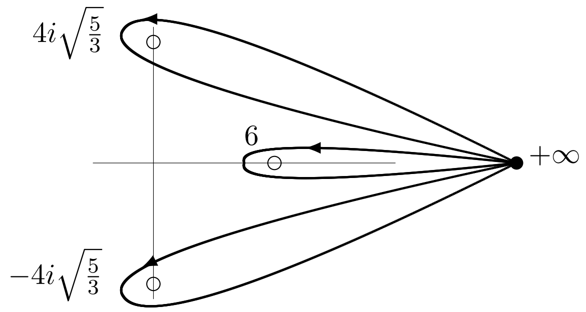
Example 2.6.5. — We end this section with a concrete example in which we compute an additive convolution, illustrating Theorem 2.6.4. Consider the polynomial

$$f(x) = x^6 + 3x^4 + 8x^2 + 6 = (x^2+1)^3 + 5(x^2+1) \in \mathbb{C}[x]$$

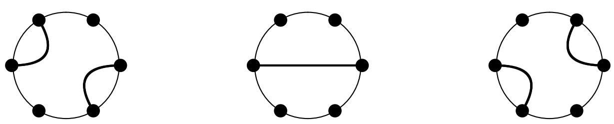
of degree 6, and regard $f$ as a potential on the variety $X = \operatorname{Spec} \mathbb{C}[x]$. Since $f\colon X \to \mathbb{A}^1$ is a finite morphism, the direct image functor $f_*$ is exact, hence $Rf_*(\underline{\mathbb{Q}}[1]) = (f_*\underline{\mathbb{Q}})[1]$. The set of singular points $S$ of the constructible sheaf $f_*\underline{\mathbb{Q}}$ is the finite set of those $s \in \mathbb{C}$ for which the polynomial $f(x) - s \in \mathbb{C}[x]$ has a root with multiplicity $> 1$. In other words, $S$ is the set of critical points $S = \{f(x) \mid f'(x) = 0\}$.

$$S = \left\{6, 4i\sqrt{\tfrac{5}{3}}, -4i\sqrt{\tfrac{5}{3}}\right\}$$

Choose loops based at $+\infty$ around the points in $S$ as follows.




The fibre of $f_*\underline{\mathbb{Q}}$ at any point $z \in \mathbb{C}$ is the vector space with basis $\{x \in \mathbb{C} \mid f(x) = z\}$. For a large real number $r$, the roots of $f(x) - r$ are in canonical bijection with the 6–th roots of unity. On these roots, monodromy along the choosen paths acts by transpositions, as follows:



Around $4i\sqrt{\frac{5}{3}}$ Around 6 Around $-4i\sqrt{\frac{5}{3}}$

If we order 6–th roots contrerclockwise $1, \zeta, \zeta^2, \zeta^3, \zeta^4, \zeta^5$ with $\zeta = \exp(2\pi i/6)$ and identify them with the canonical basis $e_1, \ldots, e_6$ of $\mathbb{Q}^6$, then the local system underlying $F$ is given by the vector space $\mathbb{Q}^6$ together with the following three automorphisms.

$$\gamma_+ = \begin{pmatrix} 0&0&0&0&0&1 \\ 0&1&0&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&0&1&0 \\ 1&0&0&0&0&0 \end{pmatrix} \qquad \gamma_0 = \begin{pmatrix} 0&0&0&1&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 1&0&0&0&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{pmatrix} \qquad \gamma_- = \begin{pmatrix} 0&1&0&0&0&0 \\ 1&0&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&0&1 \end{pmatrix}$$

Local invariants are

$$\begin{aligned} (\mathbb{Q}^6)^{\langle\gamma_+\rangle} &= \langle e_1+e_6, e_2, e_3+e_4, e_5\rangle \\ (\mathbb{Q}^6)^{\langle\gamma_0\rangle} &= \langle e_1+e_4, e_2, e_3, e_5, e_6\rangle \\ (\mathbb{Q}^6)^{\langle\gamma_-\rangle} &= \langle e_1+e_2, e_3, e_4+e_5, e_6\rangle \end{aligned}$$

whereas global invariants are the one dimensional subspace generated by $d := e_1 + e_2 + \cdots + e_6$. We see in particular that $f_*\underline{\mathbb{Q}}_X$ contains the constant sheaf $\underline{\mathbb{Q}}_{\mathbb{A}}$ as a direct factor. The sheaf $F = f_*\underline{\mathbb{Q}}_X/\underline{\mathbb{Q}}_{\mathbb{A}}$ is given by the vector space $V := \mathbb{Q}^6/d\mathbb{Q}$, the induced monodromy operators which we still denote by $\gamma_+, \gamma_0$ and $\gamma_-$, and stalks $V_+ = (\mathbb{Q}^6)^{\langle\gamma_+\rangle}/d\mathbb{Q}$, $V_0 = (\mathbb{Q}^6)^{\langle\gamma_0\rangle}/d\mathbb{Q}$ and $V_- = (\mathbb{Q}^6)^{\langle\gamma_-\rangle}/d\mathbb{Q}$. Let us check that $\Pi(f_*\underline{\mathbb{Q}}_X[-1])$ is equal to $F[-1]$. Indeed, we have $V_+ \cap V_0 \cap V_- = \{0\}$ and

$$\dim(V_+) + \dim(V_0) + \dim(V_-) = 3 + 4 + 3 = 10 = (3-1)\cdot 5 = (\#S - 1)\dim V$$

hence $R\pi_*F = 0$ and $F[-1] \in \mathrm{Perv}_0(\mathbb{A}^1_k)$. This works for all non-constant polynomials, not just the particular $f$ we have chosen.

## 2.7. Monodromic vector spaces

In this section, we introduce a local variant of the category $\mathbf{Perv}_0$, which we call category of monodromic vector spaces. It is also tannakian category. In essence, a monodromic vector space is just a vector space with an automorphism, but viewed as a perverse sheaf with vanishing cohomology on a small disk. The tensor product of monodromic vector spaces is given by additive convolution. The functor $\Psi_\infty$, as well as the total vanishing cycles functor which we shall introduce in the next section, naturally factor over monodromic vector spaces. The main result of the present section states that the category of monodromic vector spaces is equivalent, as a tannakian category, to the category of vector spaces with an automorphism.

Definition 2.7.1. — We call category of *monodromic vector spaces* and denote by $\mathbf{Vec}^\mu$ the full tannakian subcategory of $\mathbf{Perv}_0$ consisting of those objects whose only singularity is $0 \in \mathbb{C}$.

2.7.2. — Consider a perverse sheaf $F$ on a disk $D = \{z \in \mathbb{C} \mid |z| < \varepsilon\}$, and suppose that 0 is the only singularity of $F$. There is a unique way of extending $F$ to a perverse sheaf on $\mathbb{C}$ whose only singularity is $0 \in \mathbb{C}$. In this section, the distinction between such a sheaf $F$ on $D$ and its extension to the whole complex plane is irrelevant. In particular, we shall think of monodromic vector spaces as being perverse sheaves $F$ defined on some disk of unspecified but ideally small size, locally constant outside $\{0\}$ and with trivial fibre at 0.

As an abelian category, $\mathbf{Vec}^\mu$ is evidently equivalent to the category of local systems on a punctured disk, or alternatively, the category of vector spaces with an automorphism $\mathbf{Rep}(\mathbb{Z})$. Equivalences of categories inverse to each other are the functors

$$\Phi_0 : \mathbf{Vec}^\mu \to \mathbf{Rep}(\mathbb{Z}) \qquad \text{and} \qquad (-)_![1] : \mathbf{Rep}(\mathbb{Z}) \to \mathbf{Vec}^\mu$$

sending an object of $\mathbf{Vec}^\mu$ to its vanishing cycles near 0, respectively sending a representation of $\mathbb{Z}$ corresponding to a local system $L$ on $\mathbb{C} \setminus \{0\}$ to the perverse sheaf $j_!(L)[1]$, where $j\colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$ is the inclusion. Notice that, when restricted to $\mathbf{Vec}^\mu$, the functor of vanishing cycles $\Phi_0$ agrees with the functor of nearby cycles $\Psi_0$, and $\Psi_\infty$ is obtained as the composite of $\Phi_0$ with the functor $\mathbf{Rep}(\mathbb{Z}) \to \mathbf{Rep}(\mathbb{Z})$ sending a vector space with automorphism $(V, \rho)$ to $(V, \rho^{-1})$.

THEOREM 2.7.3. — *The functor of vanishing cycles* $\Phi_0\colon \mathbf{Vec}^\mu \to \mathbf{Rep}(\mathbb{Z})$ *and its inverse* $(-)_![1]\colon \mathbf{Rep}(\mathbb{Z}) \to \mathbf{Vec}^\mu$ *are equivalences of tannakian categories.*

2.7.4. — Let us spell out in detail what the statement of Theorem 2.7.3 is, and at the same time outline its proof. First of all, for any two objects $A[1]$ and $B[1]$ of $\mathbf{Vec}^\mu$, there is a canonical isomorphism of vector spaces

$$\alpha_{A,B}\colon \Phi_0(A[1]) \otimes \Phi_0(B[1]) \xrightarrow{\cong} \Phi_0(A[1] * B[1]) \tag{2.7.4.1}$$

which is functorial in $A$ and $B$. We have to construct one such isomorphism. This will be done in Proposition 2.7.5, where in fact we construct an isomorphism $\alpha_{A,B}$ for objects $A[1]$ and $B[1]$ of $\mathbf{Perv}_0$ with arbitrary singularities. Next, we have to check that $\alpha_{A,B}$ is compatible with the monodromy automorphisms on both sides of (2.7.4.1). This will be done in Proposition 2.7.7. That's not all! We have to check that our construction of $\alpha_{A,B}$ is compatible with associativity and commutativity constraints. This turns out to be a nontrivial issue. For any two objects $A[1]$ and $B[1]$ of $\mathbf{Vec}^\mu$, the diagram

$$\begin{array}{ccc} \Psi_\infty(A[1]) \otimes \Psi_\infty(B[1]) & \xrightarrow{\alpha_{A,B}} & \Psi_\infty(A[1] * B[1]) \\ \Big\downarrow \mathtt{X} & & \Big\downarrow \Psi_\infty(\mathtt{X}) \\ \Psi_\infty(B[1]) \otimes \Psi_\infty(A[1]) & \xrightarrow{\alpha_{B,A}} & \Psi_\infty(B[1] * A[1]) \end{array} \tag{2.7.4.2}$$

has to commute, where $\mathtt{X}$ are commutativity constraints, and for any three objects $A[1]$, $B[1]$ and $C[1]$ of $\mathbf{Vec}^\mu$, the diagram

$$\begin{array}{ccc} \Psi_\infty(A[1]) \otimes \Psi_\infty(B[1]) \otimes \Psi_\infty(C[1]) & \xrightarrow{\alpha_{A,B}\otimes \mathrm{id}} & \Psi_\infty(A[1] * B[1]) \otimes \Psi_\infty(C[1]) \\ \Big\downarrow \mathrm{id} \otimes \alpha_{B,C} & & \Big\downarrow \alpha_{A*B,C} \\ \Psi_\infty(A[1]) \otimes \Psi_\infty(B[1] * C[1]) & \xrightarrow{\alpha_{A,B*C}} & \Psi_\infty(A[1] * B[1] * C[1]) \end{array} \tag{2.7.4.3}$$

has to commute. In the upper left and lower right corner of (2.7.4.3), the associativity constraints for the usual tensor product of vector spaces, respectively for the additive convolution are hidden. The devil made these diagrams! In order to deal with them, we have to study for $n = 1, 2, 3$ the categories $\mathbf{C}(n)$ of sheaves on $\mathbb{A}^n$ which are constructible with respect to the stratification given by coordinate planes and describe explicitly the following functors between them:

$$\mathbf{C}(3) \underset{R(\mathrm{sum}\times\mathrm{id})_*}{\overset{R(\mathrm{id}\times\mathrm{sum})_*}{\rightrightarrows}} \mathbf{C}(2) \xrightarrow{R\mathrm{sum}_*} \mathbf{C}(1)\ .$$

PROPOSITION 2.7.5. — *For any two objects $A[1]$ and $B[1]$ of $\mathbf{Perv}_0$, there is a canonical isomorphism of rational vector spaces*

$$\alpha_{A,B}\colon \Psi_\infty(A[1]) \otimes \Psi_\infty(B[1]) \xrightarrow{\cong} \Psi_\infty(A[1] * B[1])$$

*which is functorial in $A$ and $B$.*

PROOF AND CONSTRUCTION. We construct a functorial isomorphism $\alpha_{A,B}$ as claimed, using the description of the cohomology of constructible sheaves on $\mathbb{A}^1$ in terms of cocycles. Let $S_A$ and $S_B$ denote the singular sets of $A$ and $B$ respectively. Set $\Psi_\infty(A[1]) = V_A$ and $\Psi_\infty(B[1]) = V_B$. Let $r$ be a real number bigger than the norm of any element of $S_A \cup S_B$. There are canonical and natural isomorphisms $V_A \cong A[1]_r$ and $V_B \cong B[1]_r$. Besides, we have a canonical and natural isomorphism

$$\Psi_\infty(A[1] * B[1]) \cong (A * B)[-1]_{2r} \cong H^1(\mathbb{A}^1, A \otimes \tau_{2r}^* B)$$

by the fact that the singularities of $A * B$ are contained in $S_A + S_B$ and Lemma 2.4.2. The sheaf $A \otimes \tau_{2r}^* B$ has singularities in $S = S_A \cup (2r - S_B)$, hence it is lisse at $r$ with fibre $V \otimes W$. We compute $H^1(\mathbb{A}^1, A \otimes \tau_{2r}^* B)$ in terms of cocycles. As in Section 2.2, set $G_A = \pi_1(\mathbb{C} \setminus S_A, r)$ and $G_B = \pi_1(\mathbb{C} \setminus S_B, r)$, and use the Seifert-van Kampen theorem to identify $G = \pi_1(\mathbb{C} \setminus S, r)$ with the free product $G_A * G_B$.

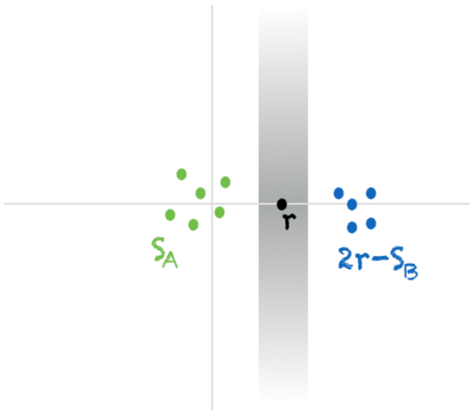



FIGURE 2.7.9. The singularities of $A \otimes \tau_{2r}^* B$

For each $s \in S_A$, let $P_{A,s}$ be the $G_A$-set of homotopy classes of paths from $s$ to $r$ in $\mathbb{C} \setminus S$, and similarly define $P_{B,s}$ for $s \in S_B$. The $G$-set $P_s$ of homotopy classes of paths from $s \in S$ to $r$ is the amalgam product $G *_{G_A} P_{A,s}$ for $s \in S_A$ and $G *_{G_B} P_{B,s}$ for $s \in 2r - S_B$. This simply means that a path from say $s \in S_A$ to $r$ can be written uniquely as a composite $gp$, where $p$ is a path from $s$ to $r$ not crossing the line $\mathrm{Re}(z) = r$, and $g \in G = G_A * G_B$ is either the empty word or a word whose last letter is not in $G_A$. We consider $G_A$ and $G_B$ as subgroups of $G$, and $P_{s,A}$ and $P_{B,s}$ as subsets of the corresponding set $P_s$.

Elements of $H^1(\mathbb{A}^1, A \otimes \tau_{2r}^* B)$ are represented by cocycles $(c, (c_s)_{s\in S})$, where $c\colon G \to V \otimes W$ and $c_s\colon P_s \to V \otimes W$ are functions satisfying $c(gh) = c(g) + gc(h)$ and $c_s(gp_s) = c(g) + gc_s(p_s)$ for all $g, h \in G$ and $p_s \in P_s$.

**Claim:** *Given any $u \in V \otimes W$, there exists a unique cocycle $(c, (c_s)_{s\in S})$ such that $c_s(p_s) = u$ for all $p_s \in P_{A,s}$ and all $s \in S_A$ and $c_s(p_s) = 0$ for all $s \in P_{B,s}$.*

Indeed, choose a path $p_s^0 \in P_{A,s}$ for each $s \in S_A$ and $p_s^0 \in P_{B,s}$ for each $s \in S_A$, and let $g_s \in G$ be the corresponding positively oriented loop around $s$. As explained in 2.2.4, there exists a unique cocycle $(c, (c_s)_{s\in S})$ with $c_s(p_s^0) = u$ for $s \in S_A$ and $c_s(p_s^0) = 0$ for $s \in 2r - S_B$. It satisfies $c(g_s) = u - g_s u$ for $s \in S_A$ and $c_s(g_s) = 0$ for $2 \in 2r - S_B$. We need to show that $c_s(p_s) = u$ for any other $p_s \in P_{A,s}$ and $c_s(p_s) = 0$ for any other $p_s \in P_{B,s}$. Since $G_A$ is generated by the $g_s$ for $s \in S_A$ and acts transitively on $P_{A,s}$, it suffices to check the case $p_s = g_t p_s^0$ for some $t \in S_A$. Indeed, $c_s(g_t p_s^0) = c(g_t) + g_t u = u$. Similary one checks that $c_s(g_t p_s) = 0$ holds for $s, t \in 2r - S_B$, and the claim follows.

Consider the map $\alpha\colon V \otimes W \to H^1(\mathbb{A}^1, A \otimes \tau_{2r}^* B)$ sending $u \in V \otimes W$ to the class of the cocycle $\alpha(u)$ given by the Claim. We show that $\alpha$ is injective: indeed, with a choice of paths $p_s^0$, the cochain complex computing the cohomology of $A \otimes \tau_{2r}^* B$ reads

$$\begin{array}{ccc} (V \otimes W) \oplus \bigoplus_{s\in S_A} (V_s \otimes W) \oplus \bigoplus_{s\in S_B} (V \otimes W_s) & \xrightarrow{d_{V\otimes W}} & \bigoplus_{s\in S_A} (V \otimes W) \oplus \bigoplus_{s\in S_B} (V \otimes W) \\ v \otimes w, (v_{A,s} \otimes w_{A,s}), (v_{B,s} \otimes w_{B,s}) & \longmapsto & (v\otimes w - p_s^0 v_{A,s} \otimes w_{A,s}), (v \otimes w - v_{B,s} \otimes p_s^0 w_{B,s}) \end{array}$$

and $\alpha(u)$ is the element $(\mathrm{diag}(u), 0)$ in the right hand space. If $\alpha(u)$ is a coboundary, then we have with the above notation

$$v \otimes w - v_{B,s} \otimes p_s^0 w_{B,s} = 0$$

for all $s \in S_B$, hence

$$v \otimes w \in \bigcap_{s\in S_B} V \otimes p_s^0 W_s = V \otimes \bigcap p_s^0 W_s = V \otimes \{0\} = \{0\}$$

holds, hence $v \otimes w = 0$. But then we find

$$-p_s^0 v_{A,s} \otimes w_{A,s} = u$$

for all for all $s \in S_A$, thus $u = 0$ by the same reasoning. This shows that $\alpha$ is injective. In order to show that $\alpha$ is surjective as well, we count dimensions. By Lemma 2.2.6, the sheaf $\mathbb{A}^1, A \otimes \tau_{2r}^* B$ has no non-zero global sections, hence we have

$$\chi(\mathbb{A}^1, A \otimes \tau_{2r}^* B) + \mathrm{rk}(A \otimes \tau_{2r}^* B) = 0 - \dim H^1(\mathbb{A}^1, A \otimes \tau_{2r}^* B) + \dim(V \otimes W) = 0$$

by Lemma 2.2.7 and the vanishing of the cohomology of $A$ and $B$.

We have thus constructed, for any $A$ and $B$, an isomorphism

$$\alpha_{A,B} : \Psi_\infty(A[1]) \otimes \Psi_\infty(B[1]) \xrightarrow{\cong} \Psi_\infty(A[1] * B[1])$$

which is functorial in $A$ and $B$. □

Corollary 2.7.6. — *For any two objects $A[1]$ and $B[1]$ of $\mathbf{Vec}^\mu$, there is a canonical isomorphism of rational vector spaces*

$$\alpha_{A,B}\colon \Phi_0(A[1]) \otimes \Phi_0(B[1]) \xrightarrow{\cong} \Phi_0(A[1] * B[1])$$

*which is functorial in $A$ and $B$.*

Proof. □

Proposition 2.7.7. — *For any two objects $A[1]$ and $B[1]$ of $\mathbf{Vec}^\mu$, the isomorphism $\alpha_{A,B}$ is compatible with monodromy operators.*

Proof. Set $A = j_!L$ and $B = j_!M$ for local systems $L$ and $M$ on $\mathbb{C} \setminus \{0\}$. The only singularity of $j_!L[1] * j_!M[1]$ is 0, hence its fibre at 0 vanishes and $j_!L[1] * j_!M[1]$ is uniquely determined by a local system on $\mathbb{A}^1 \setminus \{0\}$. We have to show that this local system is $L \otimes M$. To this end, we fix $r > 0$, and follow the recipe laid out in 2.6.2 to compute the monodromy action of $\pi_1(\mathbb{C} \setminus \{0\}, 2r)$ on $(j_!L * j_!M)[1]_{2r}$. We start with the braid action. Let $p_0$ and $p_{2r}$ be straight paths from $r \in \mathbb{C}$ to 0 and to $2r$ respectively, and let $g_0$ and $g_{2r}$ be the corresponding positively oriented generators of $G = \pi_1(\overline{\mathbb{C}} \setminus \{0, 2r\}, r)$. Let $\gamma$ be the positively oriented generator of $\pi_1(\mathbb{C} \setminus \{0\}, 2r)$. The braid

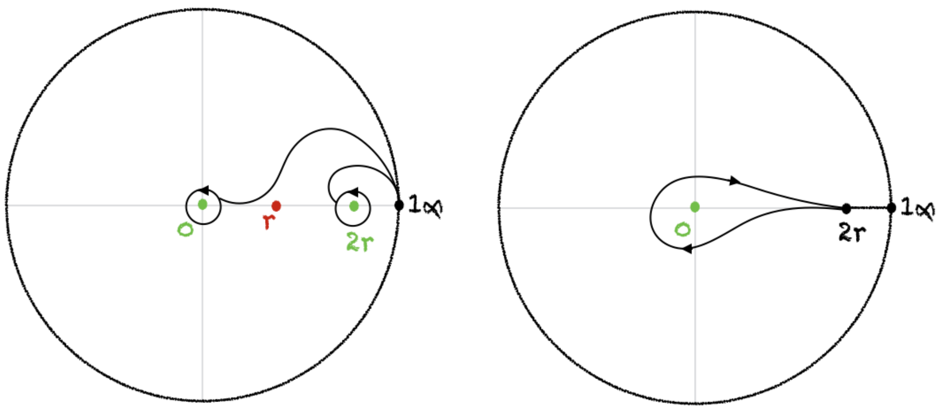



Figure 2.7.10. loops $g_0$ and $g_{2r}$ (left) and $\gamma^{-1}$ (right)

action on path spaces is given by $\beta(\gamma^{-1})(p_0) = g_{2r}p_0$ and $\beta(\gamma^{-1})(p_{2r}) = g_{2r}g_0p_{2r}$. Let us denote by $V$ and $W$ the fibres of $L$ and $M$ at $r$. The fibre at $r$ of $L \otimes \tau_{2r}^*M$ is then $V \otimes W$. We can identify the fibre

$$(j_!L * j_!M)[1]_{2r} \cong H^1(\mathbb{A}^1, j_!L \otimes \tau_{2r}^* j_!M)$$

with $V \otimes W$ via the map sending $v \otimes w$ to the class of the unique cocycle $(c, c_0, c_{2r})$ satisfying $c_0(p_0) = v \otimes w$ and $c_{2r}(p_{2r}) = 0$. This identification is the one we used to produce the isomorphism $\Psi_\infty(j_!L[1]) \otimes \Psi_\infty(j_!M[1]) \to \Psi_\infty(j_!L[1] * j_!M[1])$ in Proposition 2.7.5. According to Theorem 2.6.4, the monodromy action of $\gamma$ on $H^1(\mathbb{A}^1, j_!L \otimes \tau_{2r}^* j_!M)$ sends the class of the cocycle $c$ to the unique cocycle $(\gamma c, \gamma c_0, \gamma c_{2r})$ satisfying

$$\gamma c_0(p_0) = c_0(g_{2r}p_0) = c(g_{2r}) + g_{2r}c_0(p_0)$$

and

$$\gamma c_{2r}(p_{2r}) = c_{2r}(g_{2r}g_0p_{2r}) = c(g_{2r}) + g_{2r}c(g_0) + g_{2r}g_0c_{2r}(p_{2r}).$$

We have $v \otimes w = c(p_0) = c(g_0p_0) = c(g_0) + g_0c_0(p_0)$, hence $c(g_0) = v \otimes w - g_0v \otimes w$ and by the same reasoning $c(g_{2r}) = 0$, hence

$$\gamma c_0(p_0) = v \otimes g_{2r}w \qquad \text{and} \qquad \gamma c_{2r}(p_{2r}) = v \otimes g_{2r}w - g_0v \otimes g_{2r}w$$

Adding to $(\gamma c, \gamma c_0, \gamma c_{2r})$ the coboundary $(d, d_0, d_{2r})$ defined by $d_0(p_0) = d_{2r}(p_{2r}) = -v \otimes g_{2r}w + g_0v \otimes g_{2r}w$ shows what we want. □

Remark 2.7.8. — (1) The reader with a fondness for stars and shrieks might wonder at this point whether there is a six operations proof of Proposition 2.7.5. As it turns out, such a proof can not exist. The reason for that is that the statement of the proposition is false in other contexts with six functor formalism, for example in the framework of mixed Hodge modules.

(2) We have used $r \in \mathbb{C}$ as a base point with respect to which we computed the cohomology of $A \otimes \tau_{2r}^* B$. We could have used any other point outside $S_A \cup (2r - S_B)$ for that purpose.

(3) The construction of $\alpha_{A,B}$ is slightly asymmetric in $A$ and $B$. On two occasions we treated $A$ and $B$ differently. First, in the choice of the parametrisation $z \longmapsto (z, 2r - z)$ of the affine line $\{(x, y) \in \mathbb{A}^2 \mid x + y = 2r\} = \mathrm{sum}^{-1}(2r)$, and secondly in the definition of the cocycle $(c, c_S)$ in the claim. Interchanging the role of $A$ and $B$ in the claim leads to the isomorphism $-\alpha_{A,B}$.

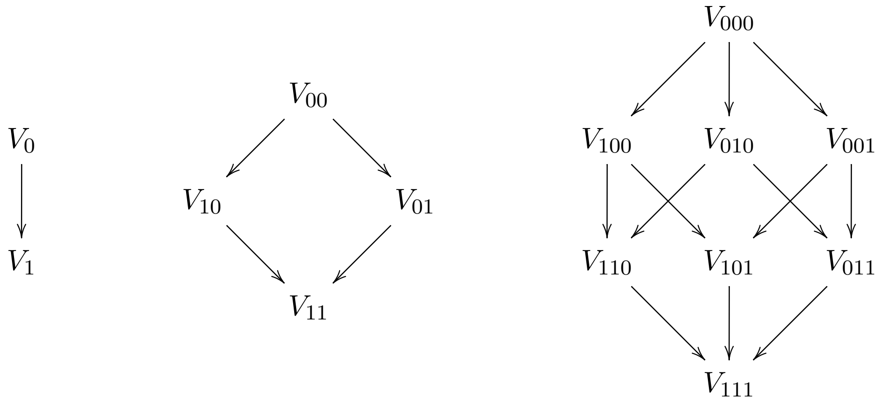
2.7.9. — For each integer $n \geqslant 1$, let $\mathbf{C}(n)$ be the category of sheaves on $\mathbb{A}^n$ which are weakly constructible with respect to the stratification given by coordinate planes. We equip the set $\{0, 1\}^n$ with its natural partial order, and denote by $\mathbb{Z}(\alpha) \subseteq \mathbb{Z}^n$ the subgroup generated by all $\beta \leqslant \alpha$. Let $I(n)$ be the category whose objects are the elements of $\{0, 1\}^n$, and whose morphisms from $\alpha$ to $\beta$ are

$$\mathrm{Mor}(\alpha, \beta) = \begin{cases} \varnothing & \beta > \alpha \\ \mathbb{Z}(\alpha)p_{\alpha\beta} & \beta \leqslant \alpha \end{cases}$$

where $p_{\alpha\beta}$ is nothing but a symbol for recovering the source and the target of a morphism. The identity of $\alpha$ is $0p_{\alpha\alpha}$, and the composition law is given by

$$(vp_{\beta\alpha})(up_{\gamma\beta}) = (u + v)p_{\gamma\alpha}$$

for $\gamma \leqslant \beta \leqslant \alpha$. To give a functor from $I(n)$ to the category of vector spaces is to give for every $\alpha \in \{0, 1\}^n$ a vector space $V_\alpha$, maps $V_\beta \to V_\alpha$ for $\beta \leqslant \alpha$, and for each non-zero coordinate of $\alpha$ an automorphism of $V_\alpha$. These automorphisms are required to commute with each other and be compatible with the maps $V_\beta \to V_\alpha$ in the appropriate way. The category $\mathbf{C}(n)$ is canonically equivalent to the category of functors from $I(n)$ to vector spaces. Constructible sheaves correspond to functors with values in finite-dimensional vector spaces. With a sheaf $F$, we associate the functor $I(n) \to \mathbf{Vec}$ given by the collection of vector spaces $V_\alpha$, where $V_\alpha$ is the fibre of $F$ at $\alpha \in \{0,1\}^n \subseteq \mathbb{C}^n$, the maps $V_\beta \to V_\alpha$ are cospecialisation maps for straight paths from $\beta$ to $\alpha$, and the automorphisms of $V_\alpha$ are the corresponding monodromy operators. In particular for $n = 1, 2, 3$, an object of $\mathbf{C}(n)$ is a diagram of vector spaces of the following shape:




(2.7.9.1)

These vector spaces $V_\alpha$ come equipped with commuting automorphisms, one for every non-zero coordinate of $\alpha$, say on $V_{110}$ we are given commuting automophisms $g_{\bullet 10}$ and $g_{1\bullet 0}$. For the given map $p_{1\bullet 0}\colon V_{100} \to V_{110}$, the relations

$$p_{1\bullet 0}\, g_{\bullet 00} = g_{\bullet 10}\, p_{1\bullet 0} \qquad \text{and} \qquad p_{1\bullet 0} = g_{1\bullet 0}\, p_{1\bullet 0}$$

must be satisfied.

2.7.10. — We shall now describe the functor $R\mathrm{sum}_*\colon \mathbf{C}(2) \to \mathbf{C}(1)$ in the combinatorial terms introduced in 2.7.9. Let $F$ be an object of $\mathbf{C}(2)$, corresponding to a commutative diagram of vector spaces and automorphisms

$$\begin{array}{ccccc} & & V_{00} & & \\ & {\scriptstyle p_{\bullet 0}}\swarrow & & \searrow{\scriptstyle p_{0\bullet}} & \\ V_{10} & & & & V_{01} \\ & {\scriptstyle p_{1\bullet}}\searrow & & \swarrow{\scriptstyle p_{\bullet 1}} & \\ & & V_{11} & & \end{array} \qquad \begin{array}{c} g_{\bullet 0} \in \mathrm{GL}(V_{10}) \quad g_{0\bullet} \in \mathrm{GL}(V_{01}) \\ \\ g_{\bullet 1}, g_{1\bullet} \in \mathrm{GL}(V_{11}) \end{array}$$

satisfying appropriate compatibilities. Set $g_{\bullet\bullet} := g_{1\bullet} g_{\bullet 1}$ and $p_{\bullet\bullet} := p_{0\bullet} p_{\bullet 1}$. We associate with it the two-term complex $S(F) := S^0(F) \to S^1(F)$ in $\mathbf{C}(1)$, where $S^0(F)$ and $S^1(F)$ correspond to the columns in the diagram

$$\begin{array}{ccc} V_{00} & \longrightarrow & 0 \\ {\scriptstyle p_\bullet}\downarrow & & \downarrow \\ V_{10} \oplus V_{01} \oplus V_{11} & \xrightarrow{\ d\ } & V_{11} \oplus V_{11} \end{array}$$

with cospecialisation $p_\bullet(v_{00}) = (p_{\bullet 0}(v_{00}), p_{0\bullet}(v_{00}), p_{\bullet\bullet}(v_{00}))$ together with the monodromy automorphisms $(g_{\bullet 0}, g_{0\bullet}, g_{\bullet\bullet})$ of $V_{10} \oplus V_{01} \oplus V_{11}$ and $(g_{\bullet\bullet}, g_{\bullet\bullet})$ of $V_{11} \oplus V_{11}$. The differential $d$ is given by

$$d(v_{10}, v_{01}, v_{11}) = (v_{11} - p_{\bullet 0}(v_{10}), v_{11} - p_{0\bullet}(v_{01}))$$

which makes the whole diagram commute, and is compatible with the automorphisms.

LEMMA 2.7.11. — *There are canonical isomorphisms $R^n\mathrm{sum}_*(F) \cong H^n(S(F))$ in $\mathbf{C}(1)$, functorial in $F$ and compatible with connecting morphisms associated with short exact sequences in $\mathbf{C}(2)$.*

PROOF. There is an obvious isomorphism $\mathrm{sum}_*(F) \cong H^0(S(F))$. To prove the lemma, it suffices to show that the functor $F \longmapsto H^0(S(F))$ is effaceable. This is indeed the case, as we can embedd $F$ into a coinduced object whose cohomology is trivial (Analogy of Schapiro's Lemma - write details) $\square$

2.7.12. — We can now reinterpret the isomorphism $\alpha_{A,B}$ and reprove Proposition 2.7.7 using Lemma 2.7.11. Given objects $A[1]$ and $B[1]$ of $\mathbf{Vec}^\mu$, the constructible sheaves $A$, $B$ and $A \boxtimes B$ correspond to objects

$$\begin{array}{c} 0 \\ \downarrow \\ V \end{array} \quad \text{and} \quad \begin{array}{c} 0 \\ \downarrow \\ W \end{array} \quad \text{and} \quad \begin{array}{ccccc} & & 0 & & \\ & \swarrow & & \searrow & \\ 0 & & & & 0 \\ & \searrow & & \swarrow & \\ & & V \otimes W & & \end{array}$$

in $\mathbf{C}(1)$ and $\mathbf{C}(2)$ respectively, with $V = \Phi_0(A[1])$ and $W = \Phi_0(B[1])$. On $V$ and $W$ we are given monodromy automorphisms $g_V$ and $g_W$, and on $V \otimes W$ we are given the two commuting automorphisms $g_{\bullet 1} = g_V \otimes \mathrm{id}_W$ and $g_{1\bullet} = \mathrm{id}_V \otimes g_W$. According to Lemma 2.7.11, $R\mathrm{sum}_*(A \boxtimes B)$ is the complex

$$\begin{array}{ccc} 0 & \longrightarrow & 0 \\ {\scriptstyle p_\bullet}\downarrow & & \downarrow \\ V \otimes W & \xrightarrow{d} & (V \otimes W) \oplus (V \otimes W) \end{array}$$

in $\mathbf{C}(1)$, where $d$ is the diagonal map, and the monodromy automorphism on $V \otimes W$ is $g_{\bullet\bullet} = g_V \otimes g_W$. The isomorphism $\alpha_{A,B}$ is in these terms the map sending $v \otimes w \in V \otimes W = \Phi_0(A[1]) \otimes \Phi_0(B[1])$ to the class of $(v \otimes w, 0 \otimes 0)$ in $\Phi_0(A[1] * B[1]) = \mathrm{coker}\, d$.

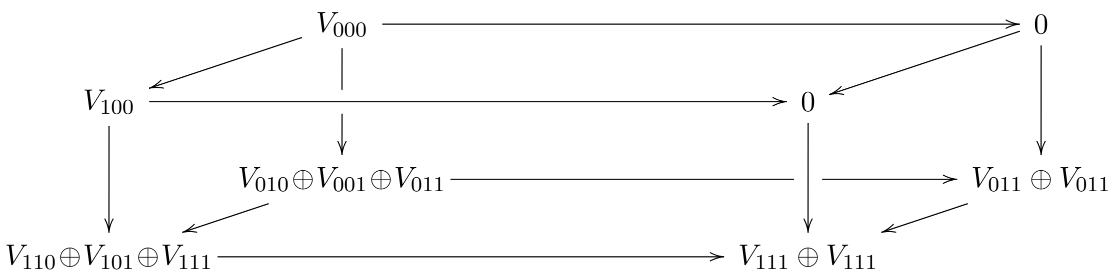
PROPOSITION 2.7.13. — *The diagram 2.7.4.2 commutes.*

PROOF. Pick objects $A[1]$ and $B[1]$ of $\mathbf{Vec}^\mu$ and set $V = \Phi_0(A[1])$ and $V = \Phi_0(B[1])$. The commutativity constraint in $\mathbf{Perv}_0$ is the isomorpism $\mathtt{x}\colon A[1] * B[1] \to B[1] * A[1]$ given by the following composition.

$$\begin{aligned} A[1] * B[1] &= R\mathrm{sum}_*(\mathrm{pr}_1^* A[1] \otimes \mathrm{pr}_2^* B[1]) \\ &= R\mathrm{sum}_*(\mathrm{pr}_2^* B[1] \otimes \mathrm{pr}_1^* A[1]) \\ &= R\mathrm{sum}_*\sigma_*(\mathrm{pr}_1^* B[1] \otimes \mathrm{pr}_2^* A[1]) \\ &= B[1] * A[1] \end{aligned}$$

□

2.7.14. — Our next task is to describe the functors $R(\mathrm{sum} \times \mathrm{id})_* : \mathbf{C}(3) \to \mathbf{C}(2)$ and also $R(\mathrm{sum}^3)_* : \mathbf{C}(3) \to \mathbf{C}(1)$ in similar terms. Fix an object of $\mathbf{C}(3)$ given by a diagram as on the far right of (2.7.9.1). We associate with it the two term complex




in $\mathbf{C}(2)$, and a three term complex

in $\mathbf{C}(1)$.

Proposition 2.7.15. — *The diagram 2.7.4.3 commutes.*

## 2.8. The fibre at infinity and vanishing cycles as fibre functors

In this section, we prove that the functor fibre near infinity $\Psi_\infty \colon \mathbf{Perv}_0 \to \mathbf{Vec}$ is a fibre functor on the tannakian category $\mathbf{Perv}_0$. We already know that $\Psi_\infty$ is faithful and exact, and it remains to show that $\Psi_\infty$ is compatible with tensor products. Besides the fibre functor $\Psi_\infty$, there is another interesting and useful fibre functor

$$\Phi \colon \mathbf{Perv}_0 \to \mathbf{Vec}_{\mathbb{Q}}$$

which sends an object $A[1]$ of $\mathbf{Perv}_0$ to the sum over $z \in \mathbb{C}$ of the space of vanishing cycles of $A$ at $z$. We obtain from $\Phi$ not just a vector space, but a $\mathbb{C}$-graded vector space, where each graded piece is equipped with an automorphism given by the local monodromy operator as was recalled in 2.1.19. That the functor $\Phi$ is compatible with the additive convolution, even if we enrich it to a functor

$$\Phi \colon \mathbf{Perv}_0 \to \{\mathbb{C}\text{-graded representations of } \mathbb{Z}\}$$

is essentially the statement of the Thom–Sebastiani theorem for functions in one variable, except that we deal here with a global version of it.

Theorem 2.8.1. — *The fibre near infinity* $\Psi_\infty\colon \mathbf{Perv}_0 \to \mathbf{Vec}$ *is a fibre functor.*

Proof. Given an object $A[1]$ of $\mathbf{Perv}_0$, let us call *monodromic fibre near infinity* the monodromic vector space

$$\Psi^\mu_\infty(A[1]) := j_! u^* F$$

where $j\colon \mathbb{C}\setminus 0 \to \mathbb{C}$ is the inclusion and $u : \mathbb{C}\setminus 0 \to \mathbb{C}$ is the function $u(ce^{i\theta}) = \max(r, c^{-1})e^{-i\theta}$ for some real $r$, larger than the norm of each singularity of $A$. The functor $\Psi_\infty$ factors as

$$\mathbf{Perv}_0 \xrightarrow{\Psi^\mu_\infty} \mathbf{Vec}^\mu \xrightarrow{\Phi_0} \mathbf{Vec}$$

and by Theorem 2.7.3, the vanishing fibre functor $\Phi_0$ is a fibre functor on the tannakian category of monodromic vector spaces $\mathbf{Vec}^\mu$. It suffices thus to show that the monodromic fibre near infinity functor $\Psi^\mu_\infty$ is compatible with tensor products. This is a six-operations exercise.

$$\begin{array}{ccccc}
\mathbb{C}\times\mathbb{C} & \xleftarrow{u\times u} & \mathbb{C}^\times\times\mathbb{C}^\times & \xrightarrow{j\times j} & \mathbb{C}\times\mathbb{C} \\
\Big\downarrow{\scriptstyle \mathrm{sum}} & & & & \Big\downarrow{\scriptstyle \mathrm{sum}} \\
\mathbb{C} & \xleftarrow{\quad u \quad} & \mathbb{C}^\times & \xrightarrow{\quad j \quad} & \mathbb{C}
\end{array}$$

We have to show that for a sheaf $A\boxtimes B$ on $\mathbb{C}\times\mathbb{C}$ in the upper left corner of the diagram, we have a natural isomorphism of sheaves

$$R\mathrm{sum}_*(j\times j)_!(u\times u)^*(A\boxtimes B) \;\cong\; j_! u^* R\mathrm{sum}_*(A\boxtimes B)$$

on $\mathbb{C}$ the lower left corner in the diagram. □

Definition 2.8.2. — We call *total vanishing cycles functor* the functor

$$\begin{array}{rcl}
\Phi\colon \mathbf{Perv}_0 & \longrightarrow & \{\mathbb{C}\text{-graded representations of } \mathbb{Z}\} \\
F & \longmapsto & \bigoplus_{z\in\mathbb{C}} \Phi_z(F)
\end{array}$$

where the $\mathbb{Z}$-action on the vanishing cycles $\Phi_z(F)$ is induced by the local monodromy on the fibre of $F$ near $z$.

Theorem 2.8.3 (Thom–Sebastiani). — *The total vanishing cycles functor is exact, faithful and monoidal: For all objects $F$ and $G$ of* $\mathbf{Perv}_0$, *there exist functorial isomorphisms*

$$\Phi(F*G) \;\cong\; \Phi(F)\otimes\Phi(G) \tag{2.8.3.1}$$

$$\Phi(F^\vee) \;\cong\; \Phi(F)^\vee \tag{2.8.3.2}$$

*in the category of $\mathbb{C}$-graded representations of $\mathbb{Z}$, which are compatible with associativity, commutativity and unit constraints.*

Corollary 2.8.4. — *The composite of $\Phi$ with the forgetful functor to vector spaces is a fibre functor on the tannakian category* $\mathbf{Perv}_0$.

2.8.5. — Before we start with the proof, let us summarise what we have to show and how we will show it. First of all, we need to check that the functor $\Phi$ is faithful and exact. This is not difficult, and part of Proposition 2.8.6 where we show that $\Psi_\infty$ and $\Phi$ are isomorphic as additive functors $\mathbf{Perv}_0 \to \mathbf{Vec}_{\mathbb{Q}}$.

The essential part of Theorem 2.8.3 is of course the existence of the isomorphisms (2.8.3.1) and (2.8.3.2). We produce these in two steps. First, we interpret vanishing cycles as objects in the tannakian category $\mathbf{Vec}_{\mathbb{Q}}^{\mu}$ of local systems on a small punctured disk with additive convolution as tensor product. The latter is equivalent to the full tannakian subcategory of $\mathbf{Perv}_0$ consisting of those objects whose only singularity is at $0 \in \mathbb{C}$. If vanishing cycles are interpreted this way, it is quite straightforward to check that the total vanishing cycles functor is monoidal. The second step consists in the use of Theorem 2.7.3, which states that the category $\mathbf{Vec}_{\mathbb{Q}}^{\mu}$ is equivalent as a tannakian cateogry to the category of representations of $\mathbb{Z}$ for the usual tensor product.

Proposition 2.8.6. — *There exists an isomorphism of additive functors between the nearby fibre at infinity* $\Psi_\infty\colon \mathbf{Perv}_0 \to \mathbf{Vec}_{\mathbb{Q}}$ *and the total vanishing cycles functor* $\Phi\colon \mathbf{Perv}_0 \to \mathbf{Vec}_{\mathbb{Q}}$. *In particular,* $\Phi$ *is faithful and exact.*

Proof. By Lemma 2.1.20, the vanishing cycles functor $\Phi_z\colon \mathbf{Perv} \to \mathbf{Vec}_{\mathbb{Q}}$ is exact for every $z \in \mathbb{C}$, and the inclusion $\mathbf{Perv}_0 \to \mathbf{Perv}$ is exact, hence $\Phi$ is exact. Fix a finite set of singularities $S \subseteq \mathbb{C}$, and denote by $\mathbf{Perv}_0(S)$ the full subcategory of $\mathbf{Perv}_0$ consisting of those objects whose singularities are contained in $S$. We can regard $\mathbf{Perv}_0(S)$ as the category of representations of $(G, P_S)$, where $G$ is the fundamental group of $\overline{\mathbb{C}} \setminus S$ based at $1\infty$, and $P_s$ is the $G$-set of paths from $s$ to $1\infty$. Let $A[1]$ be an object of $\mathbf{Perv}_0(S)$ corresponding to a representation $V = (V, (V_s)_{s\in S}, \rho, (\rho_s)_{s\in S})$. By choosing for each $s \in S$ a path $p_s^0 \in P_s$ from $s$ to $1\infty$ we can identify the nearby cycles of $A$ at $s$ with the fibres at $1\infty$, that is, with the vector space $V$. In particular, vanishing cycles are in these terms identified with

$$\Phi_s(A) \cong \operatorname{coker}(p_s^0\colon V_s \to V)$$

functorially for morphisms in $\mathbf{Perv}_0(S)$. Since the cohomology $H^*(\mathbb{A}^1, A)$ vanishes, the diagonal morphism

$$\Psi_\infty(A[1]) = V \xrightarrow{\text{diag}} \bigoplus_{s\in S} \operatorname{coker}(p_s^0\colon V_s \to V) = \Phi(A[1])$$

is an isomorphism, functorial for $A[1]$ in $\mathbf{Perv}_0(S)$. We have thus far constructed an isomorphism of functors

$$\Psi_\infty|_{\mathbf{Perv}_0(S)} \to \Phi|_{\mathbf{Perv}_0(S)} \tag{2.8.6.1}$$

from $\mathbf{Perv}_0(S)$ to $\mathbf{Vec}_{\mathbb{Q}}$ depending on the choice of a path $p_s^0 \in P_s$ for each $s \in S$. Consider now a finite set $S' \subseteq \mathbb{C}$ containing $S$, and the corresponding fundamental group $G'$ and the corresponding $G'$-sets of paths $P'_s$ for $s \in S'$. The canonical group homomorphism $G' \to G$ is surjective, and also the maps $P'_s \to P_s$ are surjective for each $s \in S$. This implies that the isomorphism of functors (2.8.6.1) can be extended to the larger subcategory $\mathbf{Perv}_0(S')$. The choice of an element

$$p_s^0 \in \lim_{s\in S} P_s$$

for each $s \in \mathbb{C}$, where the limit runs over all finite subsets $S \subseteq \mathbb{C}$ containing $s$, yields an isomorphism of additive functors $\Psi_\infty \simeq \Phi$ as claimed. $\square$

We define *monodromic vanishing cycles* at $z \in \mathbb{C}$ as the functor

$$\begin{array}{rcl}\Phi^\mu_z\colon \mathbf{Perv}_0 & \to & \mathbf{Vec}^\mu_{\mathbb{Q}} \\ A[1] & \longmapsto & \Pi_0(\delta_z^* A)\end{array}$$

where $\delta_z$ is the translation map sending a small $D$ disk around 0 to a small disk around $z$ containing no singularities of $A$ except possibly $z$, and $\Pi_0$ is the functor sending a perverse sheaf $F$ on $D$ to

$$\Pi_0(F) = \operatorname{coker}(\pi^*\pi_* F \to F)$$

where $\pi$ is the map from $D$ to a point. The total vanishing fibres functor as defined in 2.8.6 is thus the composite of functors

$$\begin{array}{rcl}\mathbf{Perv}_0 & \xrightarrow{\Phi^\mu} & \{\mathbb{C}\text{-graded monodromic vector spaces}\} \\ F & \longmapsto & \bigoplus_{z\in\mathbb{C}} \Phi^\mu_z(F)\end{array}$$

and the functor $\Phi_0$.

PROPOSITION 2.8.7. — *The total monodromic vanishing cycles functor is exact, faithful and monoidal: For all objects $A[1]$ and $B[1]$ of $\mathbf{Perv}_0$, there exist functorial isomorphisms*

$$\Phi^\mu(A[1] * B[1]) \cong \Phi^\mu(A[1]) \otimes \Phi^\mu(B[1]) \tag{2.8.7.1}$$

$$\Phi^\mu(A[1]^\vee) \cong \Phi^\mu(A[1])^\vee \tag{2.8.7.2}$$

*in the category of $\mathbb{C}$-graded monodromic vector spaces which are compatible with associativity, commutativity and unit constraints.*

PROOF. The construction of a functorial isomorphism of graded monodromic vector spaces (2.8.3.1) for objects $A[1]$ and $B[1]$ of $\mathbf{Perv}_0$ amounts to construct, for any fixed $t \in S_A + S_B$, an isomorphism of monodromic vector spaces

$$\Phi^\mu_t(A[1] * B[1]) \xrightarrow{\cong} \bigoplus_{r+s=t} \Phi^\mu_r(A[1]) \otimes \Phi^\mu_s(B[1]) \tag{2.8.7.3}$$

where the direct sum ranges over all $(r,s) \in S_A \times S_B$ with $r+s=t$. Fix a pair $(r,s) \in S_A \times S_B$ with $r+s=t$ and disks $\delta_r : D_r \to \mathbb{C}$, $\delta_s : D_s \to \mathbb{C}$ and $\delta_t : D_t \to \mathbb{C}$ centered at $r$, $s$ and $t$. By choosing $D_r$ and $D_s$ small enough, we may suppose that the box $D_{r,s} := D_r \times D_s$ is contained in $U := \operatorname{sum}^{-1}(D_t)$.

The tensor products on the right hand side of (2.8.7.3) tensor products of monodromic vector spaces, so in fact additive convolutions, namely

$$\begin{array}{rcl}\Phi^\mu_r(A[1]) \otimes \Phi^\mu_s(B[1]) & = & R\mathrm{sum}_*(\Pi_0\delta_r^* A[1] \boxtimes \Pi_0\delta_s B[1]) \\ & = & R^1\mathrm{sum}_*(\delta_{r,s}^*(A \boxtimes B))[1] \\ & = & \delta_t^* R^1\mathrm{sum}_*((A\boxtimes B)|_{D_{r,s}})[1]\,.\end{array}$$

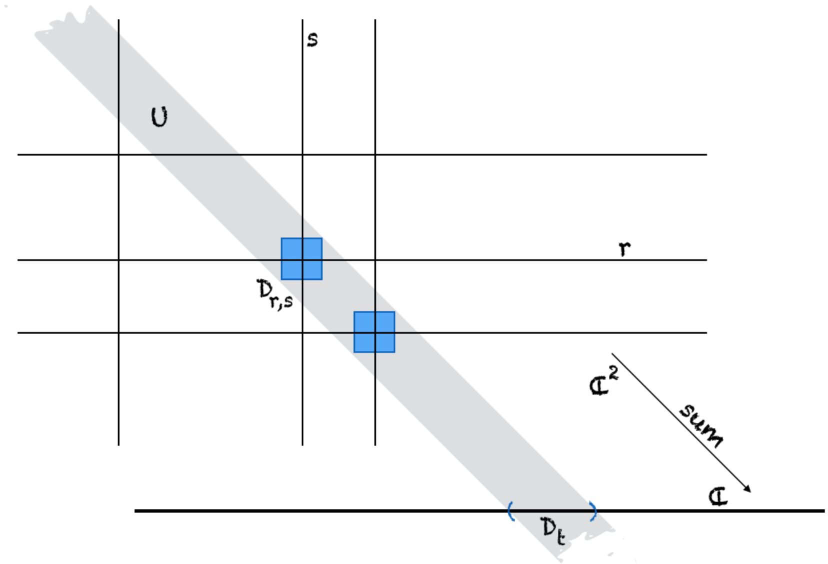



FIGURE 2.8.11. Boxes and disks

On the other hand, the monodromic vanishing cycles of $A[1] * B[1]$ near $t$ are by definition

$$\begin{aligned} \Phi_t^\mu(A[1] * B[1]) &= \Pi_0 \delta_t^* R\mathrm{sum}_*(A \boxtimes B) \\ &= \delta_t^* R^1\mathrm{sum}_*((A \boxtimes B)|_U)[1] \end{aligned}$$

so there is a natural morphism of monodromic vector spaces

$$p_{r,s}\colon \Phi_t^\mu(A[1] * B[1]) \to \Phi_r^\mu(A[1]) \otimes \Phi_s^\mu(B[1])$$

induced by the inclusion of $D_{r,s}$ into $U$. By collecting these morphisms for all $(r,s) \in S_A \times S_B$ with $r + s = t$ we obtain a morphism as displayed in (2.8.7.3), and hence a morphism

$$\Phi^\mu(A[1] * B[1]) \longrightarrow \Phi^\mu(A[1]) \otimes \Phi^\mu(B[1]) \tag{2.8.7.4}$$

of $\mathbb{C}$-graded monodromic vector spaces. This morphism is functorial in $A$ and $B$, and compatible with commutativity, associativity and unit constraints. Duals in $\mathbf{Perv}_0$ are given by $A[1]^\vee = \Pi([-1]^*\mathbb{D}(A[1]))$ as we have seen in Proposition 2.4.11, and duals in $\mathbf{Vec}_{\mathbb{Q}}^\mu$ are similarly defined using $\Pi_0$ instead of $\Pi$. We find a functorial isomorphism

$$\Phi^\mu(A[1]^\vee) \longrightarrow \Phi^\mu(A[1])^\vee \tag{2.8.7.5}$$

and it remains to show that (2.8.7.4) is an isomorphism. This follows by general tannakian nonsense from (2.8.7.5) and compatibility of (2.8.7.4) with the constraints. Indeed, an inverse to (2.8.7.5) is the composite morphism

$$\begin{aligned} \Phi^\mu(A[1]) \otimes \Phi^\mu(B[1]) &\to \Phi^\mu(A[1] * B[1] * B[1]^\vee) \otimes \Phi^\mu(B[1]) \\ &\to \Phi^\mu(A[1] * B[1]) \otimes \Phi^\mu(B[1])^\vee \otimes \Phi^\mu(B[1]) \\ &\to \Phi^\mu(A[1] * B[1]) \end{aligned}$$

where at first we use coevaluation in $\mathbf{Perv}_0$, then we use (2.8.7.4) and (2.8.7.5), and at last we use evaluation in $\mathbf{Vec}_{\mathbb{Q}}^\mu$. □

Proof of Theorem 2.8.3. All the work is done, we just have to summarise. □

## 2.9. The structure of the fundamental group

2.9.1. — The tannakian category of $\mathbb{C}$-graded representations of $\mathbb{Z}$ is easy to understand. It comes with a forgetful functor to the category of vector spaces which we take as a fibre functor. We can of course also just forget the $\mathbb{C}$-grading and keep the $\mathbb{Z}$-action, or vice-versa. The fundamental group of the category of $\mathbb{C}$-graded representations of $\mathbb{Z}$ is indeed a product

$$\pi_1\big(\{\mathbb{C}\text{-graded rep. of } \mathbb{Z}\}, \text{forget}\big) = T \times G$$

where $T$ is the fundamental group of the tannakian category of $\mathbb{C}$-graded vector spaces, and $G$ is the fundamental group of the tannakian category of vector spaces with an automorphism. The groups $T$ and $G$ can be described explicitely as follows, though this description is not particularly useful. For every finitely generated subgroup $\Gamma$ of $\mathbb{C}$, define a protorus by

$$T = \lim_{\Gamma \subseteq \mathbb{C}} \operatorname{Hom}(\Gamma, \mathbb{G}_m),$$

where the limit runs over all finitely generated subgroups $\Gamma$ of $\mathbb{C}$ ordered by inclusion, and transition maps $\operatorname{Hom}(\Gamma, \mathbb{G}_m) \to \operatorname{Hom}(\Gamma', \mathbb{G}_m)$ are given by restriction for $\Gamma' \subseteq \Gamma$. This protorus is the fundamental group of the tannakian category of $\mathbb{C}$-graded vector spaces. Alternatively, we can define $T = \operatorname{Spec} A$, where $A$ is the Hopf algebra over $\mathbb{Q}$ generated by a set of variables $\{X_z \,|\, z \in \mathbb{C}\}$, modulo the relations $X_w X_z = X_{w+z}$. These relations imply in particular $X_0 = 1$ and $X_z^n = X_{nz}$. The comultiplication is defined by $X_z \longmapsto X_z \otimes X_z$ and the antipode by $X_z \longmapsto X_{-z}$. The group $G$ is the proalgebraic completion of $\mathbb{Z}$. It can be given as a limit

$$G = \lim_{(\varphi, G_\varphi)} G_\varphi$$

running over all pairs $(\varphi, G_\varphi)$ consisting of an algebraic group $G_\varphi$ over $\mathbb{Q}$ and a group homomorphism $\varphi\colon \mathbb{Z} \to G_\varphi(\mathbb{Q})$ with Zariski dense image. Transition maps are the evident ones. Notice that, while $T$ is connected, the group of connected components of $G$ is canonically isomorphic to the profinite completion of $\mathbb{Z}$, seen as a constant affine group scheme over $\mathbb{Q}$.

Definition 2.9.2. — Let $M$ be an object of a neutral tannakian category with fibre functor $\omega$, and let $G$ be the tannakian fundamental group of $M$ acting on the vector space $V = \omega(M)$. We say that $M$ is *Lie-irreducible* or *Lie-simple* if the corresponding Lie algebra representation of $\operatorname{Lie}(G)$ on $V$ is irreducible.

2.9.3. — Let $G$ be an algebraic group with Lie algebra $\mathfrak{g}$. If a representation of $G$ is Lie irreducible, then it is irreducible, but there may exist irreducible representations of $G$ which are not Lie irreducible (*e.g.* any irreducible representation of dimension $\geqslant 2$ of a finite group). For connected groups, the notions of irreducibility and Lie irreducibility coincide. This follows from the fact that the faithful exact functor

$$\{\text{Representations of } G\} \to \{\text{Representations of } \mathfrak{g}\}$$

is full if $G$ is connected. Indeed, if $G$ is connected, then the equality $V^G = V^{\mathfrak{g}}$ holds for any representation $V$ of $G$ [**TY05**, Cor. 24.3.3], and hence the equality

$$\mathrm{Hom}_G(V, W) = (V^\vee \otimes W)^G = (V^\vee \otimes W)^{\mathfrak{g}} = \mathrm{Hom}_{\mathfrak{g}}(V, W)$$

holds for any two $G$-representations $V$ and $W$. A representation of a group $G$ is thus Lie irreducible if and only if its restriction to the connected component of the unity is irreducible. We have already classified the simple objects of $\mathbf{Perv}_0$ in Lemma 2.3.9, and now want to understand which among them are Lie-irreducible. This requires some understanding of the group of connected components of the tannakian fundamental group of $\mathbf{Perv}_0$.

Lemma 2.9.4. — *Let $F[1]$ be an object of $\mathbf{Perv}_0$ whose tannakian fundamental group is finite. Then, $0$ is the only singularity of $F$. More precisely, let $j\colon \mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1$ be the inclusion. Then, the constructible sheaf $F$ is of the form $F = j_!j^*F$, and $j^*F$ is a local system with finite monodromy on $\mathbb{A}^1 \setminus \{0\}$.*

Proof. A tannakian category has a finite fundamental group if and only if it is generated as an abelian linear category by finitely many objects. Thus, if the tannakian fundamental group of $F[1]$ is finite, there exists a finite set $S \subseteq \mathbb{C}$ containing the singularities of $F[1]$ and of all tensor constructions of $F[1]$. But if $s \in \mathbb{C}$ is a singularity of $F[1]$, then $2s$ is a singularity of the tensor square of $F[1]$, hence $2S \subseteq S$. This forces $S = \{0\}$ as required, and in particular $j^*F$ is a local system and $F = j_!j^*F$ holds. It remains to show that the local system $j^*F$ has finite monodromy. Indeed, the functor

$$j_!\colon \{\text{Local systems on } \mathbb{A}^1 \setminus \{0\}\} \to \mathbf{Perv}_0$$

is fully faithful and compatible with tensor products and duals for the usual tensor structure on local systems by Theorem 2.7.3. The tannakian fundamental group of $F[1]$ is thus the same as the tannakian fundamental group of the local system $j^*F$, which in turn is finite if and only if the monodromy operator of $j^*F$ around $\{0\}$ has finite order. □

Theorem 2.9.5. — *Let $G$ be the tannakian fundamental group of the category $\mathbf{Perv}_0$, and denote by $G^0 \subseteq G$ the connected component of the unity. There is a canonical short exact sequence*

$$1 \to G^0 \to G \to \widehat{\mathbb{Z}} \to 0,$$

*where $\widehat{\mathbb{Z}} = \lim \mathbb{Z}/n\mathbb{Z}$ is the profinite completion of $\mathbb{Z}$, viewed as a constant group scheme over $\mathbb{Q}$.*

Proof. This follows from Lemma 2.9.4 and general tannakian formalism. It suffices to observe that in any tannakian category $\mathbf{T}$ with tannakian fundamental group $G$, the full subcategory $\mathbf{T}_0$ consisting of those objects which have finite fundamental groups is a tannakian subcategory, corresponding to representations of $G/G^0$. Lemma 2.9.4 states that objects of $\mathbf{Perv}_0$ with finite tannakian fundamental group are local systems on $\mathbb{A}^1 \setminus \{0\}$ with finite monodromy, or equivalently,

$\mathbb{Q}$-linear representations of $\mathbb{Z}$ with finite image. The tannakian fundamental group of the category of $\mathbb{Q}$-linear representations of $\mathbb{Z}$ with finite image is $\widehat{\mathbb{Z}}$. □

THEOREM 2.9.6. — *Let $G$ be the tannakian fundamental group of the category $\mathbf{Perv}_0$ and let $V$ be a finite-dimensional representation of $G$. Then $H^n(G, V) = 0$ for all $n \geqslant 2$. In other words, the cohomological dimension of $G$ is 1.*

PROOF. Let $A[1]$ be an object of $\mathbf{Perv}_0$, corresponding to a representation $V$ of $G$. The subspace of fixed vectors $V^G \subseteq V$ then corresponds via tannakian duality to the subobject $\pi^*\pi_*j_*j^*A[1]$ of $A[1]$, where

$$\Pi\colon \mathbb{A}^1 \to \operatorname{Spec} k \qquad \text{and} \qquad j\colon \mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1$$

are the structure morphism and the inclusion. Therefore, since $\pi^*$ and $j^*$ are exact, the cohomological dimension of the functor $V \longmapsto V^G$ is the same as the cohomological dimension of the functor $(\pi \circ j)_*$, which is equal to 1 by Artin's vanishing theorem 2.1.11 and because $\mathbb{A}^1 \setminus \{0\}$ is an affine variety of dimension 1. □

2.9.7. — From Theorem 2.8.3 we obtain a morphism of affine group schemes

$$\Phi^*\colon \pi_1(\{\mathbb{C}\text{-graded rep. of } \mathbb{Z}\}) \longrightarrow \pi_1(\mathbf{Perv}_0, \Phi)$$

and we want to understand its kernel and its image.

PROPOSITION 2.9.8. — *The morphism $\Phi^*$ is a closed immersion.*

LEMMA 2.9.9. — *Let $V$ be a $\mathbb{C}$-graded representation of $\mathbb{Z}$. There exists an object $F$ of $\mathbf{Perv}_0$ such that $\Phi(F)$ is isomorphic to $V$ as $\mathbb{C}$-graded representation of $\mathbb{Z}$.*

PROOF. We assume without loss of generality that $V$ is indecomposable, hence pure of some degree $z \in \mathbb{C}$ for the $\mathbb{C}$-grading. In other words, we are given a complex number $z$, a vector space $V$ and an automorphism $\mu : V \to V$, and it suffices to construct an object $F$ of $\mathbf{Perv}_0$ whose space of vanishing cycles at $z$ is isomorphic to $V$, as a vector space with automorphism. There is indeed a canonical choice for $F$, namely

$$F = (j_z)_! L[1]$$

where $j_z : \mathbb{C} \setminus \{z\} \to \mathbb{C}$ is the inclusion, and $L$ is the local system on $\mathbb{C} \setminus \{z\}$ with general fibre $V$ and monodromy $\mu$ around $z$. □

PROOF OF PROPOSITION 2.9.8. The morphism $\Phi^*$ is a closed immersion if and only if every $\mathbb{C}$-graded representation of $\mathbb{Z}$ is a subquotient of $\Phi(F)$ for some $F \in \mathbf{Perv}_0$, which is a direct consequence of Lemma 2.9.9. □

DEFINITION 2.9.10. — We call an object of a neutral tannakian category *abelian* if its tannakian fundamental group is commutative.

PROPOSITION 2.9.11. — *An object $F$ of $\mathbf{Perv}_0$ is abelian if and only if it is isomorphic to a direct sum of objects with only one singularity.*

PROOF. An object of $\mathbf{Perv}_0$ which has only one singularity is of the form $E(z) \otimes F$ where $F$ is an object whose only singularity is at $0 \in \mathbb{C}$. In other words, $F$ is a monodromic vector space, corresponding to a representation $\mathbb{Z} \to \mathrm{GL}_n(\mathbb{Q})$. The fundamental group of $F$ is the Zariski closure of the image of this representation, hence $F$ is abelian. The fundamental group of $E(z)$ is trivial if $z = 0$ and $\mathbb{G}_m$ otherwise, hence $E(z)$ is abelian. One implication of the proposition follows thus from the general observation that in whatever neutral tannakian category, any tensor construction of abelian objects is abelian. Conversely, let $F$ be an abelian object in $\mathbf{Perv}_0$, so the fundamental group $G = G_F$ of $F$ is a commutative algebraic group over $\mathbb{Q}$. Write $S$ for the set of singularities of $F$, and $T_S$ for the split torus dual to the group $\mathbb{Z}S$ generated by the set of complex numbers $S$. The object $F$ corresponds to a finite-dimensional, faithful representation $V$ of $G_F$. We can decompose the vector space $V$ into eigenspaces

$$V = \bigoplus_{s \in S} V_s$$

for the action of the torus $T_S$. The eigenspace corresponding to an element in $\mathbb{Z}S$ is zero unless it belongs to $S$. Since $G_F$ is commutative, hence in particular $T_S$ is central in $G_F$, this decomposition is compatible with the action of $G_F$ on $V$. In other words, the above eigenspace decomposition is a decomposition

$$F = \bigoplus_{s \in S} F_s$$

of $F$ into a direct sum, where each summand $F_s$ has only one singularity $s \in S$. □

CHAPTER 3

# Three points of view on rapid decay cohomology

In this chapter, we present three different constructions of rapid decay cohomology. We first repeat the elementary definition from the introduction and give a few examples. Next, we associate with a tuple $[X, Y, f, n, i]$ an object of the category $\mathbf{Perv}_0$ whose fibre at infinity is the rapid decay cohomology group $H^n_{\mathrm{rd}}(X, Y, f)(i)$. This enables us to derive the statement analogous to Nori's basic lemma for rapid decay cohomology from Beilinson's basic lemma for perverse sheaves. It is a key ingredient in the definition of a tensor product for exponential motives. In the case where $X$ is a smooth variety and $Y$ a normal crossing divisor, we express rapid decay cohomology as usual relative cohomology, without any limiting process, on the real blow-up of a good compactification, a point of view which will be useful to prove the comparison isomorphism between rapid decay and de Rham cohomology in Chapter 7. Finally, we introduce cup products, the Künneth formula and Poincaré–Verdier duality for rapid decay cohomology.

## 3.1. Elementary construction

For each real number $r$, let $S_r$ be the closed complex half-plane $S_r = \{z \in \mathbb{C} \mid \mathrm{Re}(z) \geqslant r\}$. Throughout this section, all homology and cohomology groups are understood to be singular homology and cohomology with rational coefficients.

Definition 3.1.1. — Let $X$ be a complex variety, $Y \subseteq X$ a closed subvariety, and $f$ a regular function on $X$. For each integer $n \geqslant 0$, the *rapid decay homology* group in degree $n$ of the triple $[X, Y, f]$ is defined as the limit

$$H_n^{\mathrm{rd}}(X, Y, f) = \lim_{r\to+\infty} H_n(X, Y \cup f^{-1}(S_r)). \tag{3.1.1.1}$$

The limit is taken in the category of $\mathbb{Q}$-vector spaces, with respect to the transition maps induced by the inclusions $f^{-1}(S_{r'}) \subseteq f^{-1}(S_r)$ for $r' \geqslant r$. Dually, the *rapid decay cohomology* group in degree $n$ of $[X, Y, f]$ is the colimit

$$H^n_{\mathrm{rd}}(X, Y, f) = \operatorname*{colim}_{r\to+\infty} H^n(X, Y \cup f^{-1}(S_r)) \tag{3.1.1.2}$$

with respect to the transition maps induced by the same inclusions. Whenever the subvariety $Y$ is empty, we shall drop it from the notation.

3.1.2. — Given $X$, $Y$, and $f$ as in 3.1.1, there exists a real number $r_0$ such that, for all $r \geqslant r_0$ and all $z \in S_r$, the inclusions

$$Y \cup f^{-1}(z) \;\subseteq\; Y \cup f^{-1}(S_r) \;\subseteq\; Y \cup f^{-1}(S_{r_0}) \tag{3.1.2.1}$$

are homotopy equivalences, hence the transition maps in (3.1.1.1) and (3.1.1.2) are eventually isomorphisms. Indeed, it follows from resolution of singularities and Ehresmann's fibration theorem (see *e.g.* [**Voi02**, Proposition 9.3]) that, given any morphism of complex algebraic varieties, in our case $f\colon X \to \mathbb{A}^1$, there exists a non-empty Zariski open subset $U$ of the target space such that $f^{-1}(U) \to U$ is a fibre bundle for the complex topology (see also [**Ver76**, Corollaire 5.1]). Together with the well-known fact that complex algebraic varieties admit a finite triangulation, this implies that rapid decay homology and cohomology groups are finite-dimensional vector spaces dual to each other. They depend naturally on $[X, Y, f]$ for the evident notion of morphisms of pairs of varieties with function. If $f$ is constant, we recover the usual singular homology of the pair $[X, Y]$, since in that case the set $f^{-1}(S_r)$ is empty for sufficiently large $r$. For large enough real $r$, we obtain isomorphisms

$$H^n_{\mathrm{rd}}(X, Y, f) \cong H^n(X, Y \cup f^{-1}(z)) \tag{3.1.2.2}$$

for any $z \in \mathbb{C}$ with $\mathrm{Re}(z) \geqslant r$. The isomorphism (3.1.2.2) is natural, in the sense that if we are given a finite family of pairs of varieties with potential $(X_\alpha, Y_\alpha, f_\alpha)$ and morphisms between them, then for any $z \in \mathbb{C}$ with sufficiently large real part the isomorphism (3.1.2.2) is natural with respect to the given morphisms.

Example 3.1.3. — Let $X = \mathbb{A}^2 = \operatorname{Spec} \mathbb{C}[x, y]$, together with the regular function $f(x, y) = x^a y^b$ for two integers $a, b \geqslant 1$. If $a$ and $b$ are coprime, the curve $f^{-1}(r) \subseteq X$ is isomorphic to $\mathbb{G}_m$ as long as $r \neq 0$. In general, if $d$ denotes the greatest common divisor of $a$ and $b$, the subvariety $f^{-1}(r)$ is a finite disjoint union of copies of $\mathbb{G}_m$ indexed by the group of roots of unity $\mu_d(\mathbb{C})$. From the long exact sequence of relative cohomology

$$0 \to H^0(\mathbb{A}^2) \to H^0(f^{-1}(r)) \to H^1(\mathbb{A}^2, f^{-1}(r)) \to H^1(\mathbb{A}^2) \to H^1(f^{-1}(r)) \to H^2(\mathbb{A}^2, f^{-1}(r)) \to 0,$$

it follows that $\dim H^1_{\mathrm{rd}}(X, f) = d - 1$ and $\dim H^2_{\mathrm{rd}}(X, f) = d$.

Example 3.1.4. — Set $X = \mathbb{G}_m^2 = \operatorname{Spec} \mathbb{C}[u^{\pm 1}, v^{\pm 1}]$, and $f(u, v) = u + v + (uv)^{-1}$. In order to compute the rapid decay homology of $(X, f)$, we need to understand the topology of the hypersurface $f(u, v) = r$ for large real $r$. Let us look at $X$ as the zero locus of $xyz - 1$ in $\mathbb{A}^3 = \operatorname{Spec} \mathbb{C}[x, y, z]$ via $(u, y) \longmapsto (u, v, (uv)^{-1})$ and extend $f$ to the function $f(x, y, z) = x + y + z$ on $\mathbb{A}^3$. The equation $f(x, y, z) = r$ describes a hypersurface in $\mathbb{A}^3$ which has the homotopy type of a honest sphere $S^2$, while $xyz = 1$ has the homotopy type of a torus $S^1 \times S^1$. Finish this

3.1.5. — Let $X$ be a complex variety, $Z \subseteq Y \subseteq X$ closed subvarieties and $f$ a regular function on $X$. As in ordinary singular cohomology, there is a canonical long exact sequence

$$\cdots \to H^{n-1}_{\mathrm{rd}}(Y, Z, f_{|Y}) \to H^n_{\mathrm{rd}}(X, Y, f) \to H^n_{\mathrm{rd}}(X, Z, f) \to \cdots \tag{3.1.5.1}$$

which is functorial in $[X, Y, Z, f]$. It is obtained as follows: for each real number $r$, there is a natural long exact sequence of cohomology groups

$$\cdots \to H^{n-1}(Y \cup f^{-1}(S_r), Z \cup f^{-1}(S_r)) \to H^n(X, Y \cup f^{-1}(S_r)) \to H^n(X, Z \cup f^{-1}(S_r)) \to \cdots$$

associated to the triple of topological spaces $Z \cup f^{-1}(S_r) \subseteq Y \cup f^{-1}(S_r) \subseteq X$. The inclusion of pairs $[Y, f|_Y^{-1}(S_r)] \to [Y \cup f^{-1}(S_r), Z \cup f^{-1}(S_r)]$ induces an isomorphism in relative cohomology by excision. We may hence identify the previous long exact sequence with

$$\cdots \to H^{n-1}(Y, Z \cup f|_Y^{-1}(S_r)) \to H^n(X, Y \cup f^{-1}(S_r)) \to H^n(X, Z \cup f^{-1}(S_r)) \to \cdots$$

and obtain (3.1.5.1) by passing to the limit.

EXAMPLE 3.1.6. — Let $X$ be a variety, $Y \subseteq X$ a closed subvariety, and $f\colon X \times \mathbb{A}^1 \to \mathbb{A}^1$ the projection onto the second factor. All the rapid decay cohomology groups vanish:

$$H^n_{\mathrm{rd}}(X \times \mathbb{A}^1, Y \times \mathbb{A}^1, f) = 0.$$

Indeed, for each real number $r$, the inclusion $f^{-1}(S_r) = X \times S_r \subseteq X \times \mathbb{C}$ is a homotopy equivalence, hence $H^n_{\mathrm{rd}}(X \times \mathbb{A}^1, f) = 0$ for all $n$. Similarly, $H^n_{\mathrm{rd}}(Y \times \mathbb{A}^1, f|_Y) = 0$, and the claim follows from the long exact sequence (3.1.5.1) associated to the triple $\varnothing \subseteq Y \subseteq X$.

## 3.2. Rapid decay cohomology in terms of perverse sheaves

In this section, we give a less elementary construction of rapid decay cohomology, now in terms of perverse sheaves. It has the advantage of automatically endowing $H^n_{\mathrm{rd}}(X, Y, f)$ with rich additional structure, and also of being a a construction purely in terms of the six-functors formalism, hence portable to other contexts. Ultimately, we wish to equip $H^n_{\mathrm{rd}}(X, Y, f)$ with the data of an exponential mixed Hodge structure, which is a special kind of mixed Hodge module on the complex affine line.

DEFINITION 3.2.1. — Let $X$ be a complex algebraic variety, let $Y \subseteq X$ be a closed subvariety, and let $f\colon X \to \mathbb{A}^1$ be a regular function on $X$. We call the object

$$H^n_{\mathrm{perv}}(X, Y, f) = \Pi({}^p\mathcal{H}^n(Rf_*\underline{\mathbb{Q}}_{[X,Y]})$$

of $\mathbf{Perv}_0$ the *perverse cohomology* of $(X, Y, f)$ in degree $n$. Here, ${}^p\mathcal{H}^n$ means homology in degree $n$ with respect to the perverse $t$-structure, and $\mathbb{Q}_{[X,Y]}$ stands for the sheaf $\beta_!\beta^*\mathbb{Q}_X$ on $X$, where $\beta\colon X \setminus Y \to X$ is the inclusion.

PROPOSITION 3.2.2. — *Let $X$ be a complex algebraic variety, let $Y \subseteq X$ be a closed subvariety, and let $f\colon X \to \mathbb{A}^1$ be a regular function on $X$. Let $\Gamma \subseteq X \times \mathbb{A}^1$ be the graph of $f$, and let*

*$p : X \times \mathbb{A}^1 \to \mathbb{A}^1$ be the projection. There is a canonical and natural isomorphism in the derived category of constructible sheaves on $\mathbb{A}^1$ :*

$$\Pi(Rf_*\underline{\mathbb{Q}}_{[X,Y]}) \xrightarrow{\cong} Rp_*\underline{\mathbb{Q}}_{[X\times\mathbb{A}^1,(Y\times\mathbb{A}^1)\cup\Gamma]}[1]. \tag{3.2.2.1}$$

PROOF. Let $s : X \to (Y \times \mathbb{A}^1) \cup \Gamma$ be the morphism of algebraic varieties given by $s(x) = (x, f(x))$. It sends $Y$ to $Y \times \mathbb{A}^1$ and satisfies $p \circ s = f$, hence induces a morphism in the derived category of constructible sheaves

$$Rp_*\underline{\mathbb{Q}}_{[(Y\times\mathbb{A}^1)\cup\Gamma,Y\times\mathbb{A}^1]} \longrightarrow Rf_*\underline{\mathbb{Q}}_{[X,Y]} \tag{3.2.2.2}$$

which is natural in $[X, Y, f]$. This morphism is an isomorphism. Indeed, the map $s$ is an isomorphism of $[X, Y]$ onto its image $[\Gamma, (Y \times \mathbb{A}^1) \cap \Gamma]$, and the cohomology of this pair over any open subset of $\mathbb{A}^1(\mathbb{C})$ is isomorphic to that of $[(Y \times \mathbb{A}^1) \cup \Gamma, Y \times \mathbb{A}^1]$ by excision: cut out the open subspace $\{(y, t) \in Y \times \mathbb{A}^1 \mid f(y) \neq t\}$. From the triple of spaces $Y \times \mathbb{A}^1 \subseteq (Y \times \mathbb{A}^1) \cup \Gamma \subseteq X \times \mathbb{A}^1$ we obtain the following natural exact triangle.

$$Rp_*\underline{\mathbb{Q}}_{[X\times\mathbb{A}^1,(Y\times\mathbb{A}^1)\cup\Gamma]} \to Rp_*\underline{\mathbb{Q}}_{[X\times\mathbb{A}^1,Y\times\mathbb{A}^1]} \to Rp_*\underline{\mathbb{Q}}_{[(Y\times\mathbb{A}^1)\cup\Gamma,Y\times\mathbb{A}^1]} \to Rp_*\underline{\mathbb{Q}}_{[X\times\mathbb{A}^1,(Y\times\mathbb{A}^1)\cup\Gamma]}[1].$$

The object $Rp_*\underline{\mathbb{Q}}_{[X\times\mathbb{A}^1,Y\times\mathbb{A}^1]}$ is the same as $\pi^*R\pi_*\underline{\mathbb{Q}}_{[X,Y]}$, hence applying $\Pi$ to it returns the zero object. We find therefore a natural isomorphism

$$\Pi(Rf_*\underline{\mathbb{Q}}_{[X,Y]}) \xrightarrow{\cong} \Pi(Rp_*\underline{\mathbb{Q}}_{[X\times\mathbb{A}^1,(Y\times\mathbb{A}^1)\cup\Gamma]}[1])$$

and it remains to show that the adjunction $Rp_*\underline{\mathbb{Q}}_{[X\times\mathbb{A}^1,(Y\times\mathbb{A}^1)\cup\Gamma]} \to \Pi(Rp_*\underline{\mathbb{Q}}_{[X\times\mathbb{A}^1,(Y\times\mathbb{A}^1)\cup\Gamma]})$ is an isomorphism. The triangle (2.4.4.2) reads

$$\pi^*R\pi_*Rp_*\underline{\mathbb{Q}}_{[X\times\mathbb{A}^1,(Y\times\mathbb{A}^1)\cup\Gamma]} \to Rp_*\underline{\mathbb{Q}}_{[X\times\mathbb{A}^1,(Y\times\mathbb{A}^1)\cup\Gamma]} \to \Pi(Rp_*\underline{\mathbb{Q}}_{[X\times\mathbb{A}^1,(Y\times\mathbb{A}^1)\cup\Gamma]})$$

hence we must show that $R(\pi \circ p)_*\underline{\mathbb{Q}}_{[X\times\mathbb{A}^1,(Y\times\mathbb{A}^1)\cup\Gamma]}$ is zero. This is just a complicated way of saying that the cohomology groups $H^n(X \times \mathbb{A}^1, (Y \times \mathbb{A}^1) \cup \Gamma)$ vanish. The cohomology groups of the pair $[X \times \mathbb{A}^1, \Gamma]$ are zero, because this pair is homotopic to $[X, X]$. The long exact sequence of the triple $\Gamma \subseteq (Y \times \mathbb{A}^1) \cup \Gamma \subseteq X \times \mathbb{A}^1$ shows that it is enough to prove that the cohomology groups of the pair $[(Y \times \mathbb{A}^1) \cup \Gamma, \Gamma]$ vanish. The excision isomorphism shows that the cohomology of $[(Y \times \mathbb{A}^1) \cup \Gamma, \Gamma]$ is the same as the cohomology of $[Y \times \mathbb{A}^1, \Gamma \cap (Y \times \mathbb{A}^1)]$, and since $\Gamma \cap (Y \times \mathbb{A}^1)$ is just the graph of the restriction of $f$ to $Y$, this cohomology vanishes as we wanted to show. □

COROLLARY 3.2.3. — *Let $X$ be a complex variety, $Y \subseteq X$ a closed subvariety, and $f$ a regular function on $X$. There is a canonical and natural isomorphism of $\mathbb{Q}$-vector spaces*

$$\Psi_\infty(H^n_{\mathrm{perv}}(X, Y, f)) \cong H^n_{\mathrm{rd}}(X, Y, f).$$

PROOF. By Lemma 2.3.3 and part (2) of Proposition 2.4.4, given any object $C$ of the derived category of constructible sheaves, we have canonical and natural isomorphisms in $\mathbf{Perv}_0$

$$\Pi({}^p\mathcal{H}^n(C)) \cong {}^p\mathcal{H}^n(\Pi(C)) \cong \mathcal{H}^{n-1}(\Pi(C))[1].$$

Proposition 3.2.2 yields therefore an isomorphism

$$\Pi({}^p\mathcal{H}^n(Rf_*\underline{\mathbb{Q}}_{[X,Y]})) \cong \mathcal{H}^{n-1}(\Pi(Rf_*\underline{\mathbb{Q}}_{[X,Y]}))[1] \xrightarrow{\cong} \mathcal{H}^n(Rp_*\underline{\mathbb{Q}}_{[X\times\mathbb{A}^1,(Y\times\mathbb{A}^1)\cup\Gamma]})[1]$$

in the category $\mathbf{Perv}_0$. The fibre of the constructible sheaf $\mathcal{H}^n(Rp_*\underline{\mathbb{Q}}_{[X\times\mathbb{A}^1,(Y\times\mathbb{A}^1)\cup\Gamma]})$ over a large real number $r$ is the cohomology of the pair $[X\times\{r\},(Y\times\{r\})\cup(f^{-1}(r)\times\{r\})]$ in degree $n$, which is the rapid decay cohomology $H^n_{\mathrm{rd}}(X,Y,f)$ as $r$ tends to infinity. □

3.2.4. — The following will be useful later. Given a variety $X$ and a regular function $f\colon X\to\mathbb{A}^1$, we denote by $\Gamma_f$ the functor from $\mathrm{Sh}(X)$ to $\mathrm{Vec}_{\mathbb{Q}}$ obtained by composition of the following:

$$\mathrm{Sh}(X)\xrightarrow{f_*}\mathrm{Sh}(\mathbb{A}^1)\xrightarrow{F\longmapsto F\boxtimes E(0)}\mathrm{Sh}(\mathbb{A}^2)\xrightarrow{\mathrm{sum}_*}\mathrm{Sh}(\mathbb{A}^1)\xrightarrow{\psi_\infty}\mathrm{Vec}_{\mathbb{Q}}. \qquad (3.2.4.1)$$

The functor $\Gamma_f$ is left exact and its right derived functor computes rapid decay cohomology in the following sense:

Proposition 3.2.5. — *There is a canonical isomorphism*

$$H^n_{\mathrm{rd}}(X,Y,f)=R^n\Gamma_f\underline{\mathbb{Q}}_{[X,Y]}. \qquad (3.2.5.1)$$

Proof. Let $I_\bullet=[I_0\to I_1\to\cdots]$ be an injective resolution of the sheaf $\underline{\mathbb{Q}}_{[X,Y]}$ on $X$. We consider the following natural morphism of complexes of sheaves on $\mathbb{A}^1$:

$$\mathrm{sum}_*(f_*(I_\bullet)\boxtimes j_!\underline{\mathbb{Q}}_{\mathbb{G}_m})\to R\mathrm{sum}_*(f_*(I_\bullet)\boxtimes j_!\underline{\mathbb{Q}}_{\mathbb{G}_m}). \qquad (3.2.5.2)$$

By Corollary 3.2.3, the rapid decay cohomology groups $H^n_{\mathrm{rd}}(X,Y,f)$ are obtained by applying $\Psi_\infty$ to the right hand side complex, whereas applying $\Psi_\infty$ to the complex on the left hand side complex yields $R\Gamma_f(\underline{\mathbb{Q}}_{[X,Y]})$. The sheaves $f_*(I_p)$ are flasque, hence, in order to show that (3.2.5.2) is an isomorphism, it suffices to prove that flasque sheaves are acyclic for the left exact functor $F\longmapsto\mathrm{sum}_*(F\boxtimes j_!\underline{\mathbb{Q}}_{\mathbb{G}_m})$. Indeed, from the exact triangle (2.4.4.2) we obtain a long exact sequence starting with

$$0\to\mathrm{sum}_*(F\boxtimes j_!\underline{\mathbb{Q}}_{\mathbb{G}_m})\to\pi^*\pi_*(F)\to F\to R^1\mathrm{sum}_*(F\boxtimes j_!\underline{\mathbb{Q}}_{\mathbb{G}_m})\to\pi^*R^1\pi_*(F)\to 0$$

and containing isomorphisms $\pi^*R^p\pi_*(F)\cong R^p\mathrm{sum}_*(F\boxtimes j_!\underline{\mathbb{Q}}_{\mathbb{G}_m})$ for $p\geqslant 2$, where $\pi$ is the map from $\mathbb{A}^1$ to a point. If $F$ is flasque, then $R^p\pi_*(F)=0$ for all $p\geqslant 1$, and the map $\pi^*\pi_*(F)\to F$ is surjective, hence $R^p\mathrm{sum}_*(F\boxtimes j_!\underline{\mathbb{Q}}_{\mathbb{G}_m})=0$ for $p\geqslant 1$ as we wanted. □

## 3.3. Cell decomposition and the exponential basic lemma

In this section, we prove the analogue of Nori's basic lemma for rapid decay cohomology. Using its description in terms of perverse sheaves, it will be a more or less straightforward consequence of the most general version of the basic lemma, obtained by Beilinson in [**Bei87**, Lemma 3.3]. We recall the argument for the convenience of the reader.

Theorem 3.3.1 (Beilinson's basic lemma). — *Let $f\colon X \to S$ be a morphism between quasi-projective varieties over $k$, and let $F$ be a perverse sheaf on $X$. There exists a dense open subvariety $j\colon U \hookrightarrow X$ such that the perverse sheaves ${}^p\mathcal{H}^n(Rf_*j_!j^*F)$ on $S$ vanish for all $n < 0$.*

Proof. We will show that the complement in $X$ of a general hyperplane section has the desired property. To alleviate notations, we agree that for the duration of this proof all direct and inverse image functors are between derived categories of sheaves, and write them just as $f_*$ and $f_!$ instead of $Rf_*$ and $Rf_!$. Consider a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & S \\ {\scriptstyle i_X}\downarrow & & \downarrow{\scriptstyle i_S} \\ \overline{X} & \xrightarrow{\overline{f}} & \overline{S}, \end{array}$$

where $i_S$ and $i_X$ are open immersions and $\overline{S}$ and $\overline{X}$ are projective. Choose an embedding of $\overline{X}$ into some projective space $\mathbb{P} = \mathbb{P}^N_k$. Let $\mathbb{P}' \simeq \mathbb{P}^N_k$ be the dual projective space parametrising hyperplanes in $\mathbb{P}$. The family of all hyperplane sections of $\overline{X}$ is the closed subvariety of $\overline{X} \times \mathbb{P}'$ defined as

$$\overline{H} = \{(x, L) \in \overline{X} \times \mathbb{P}' \mid x \in L\}.$$

Setting $H = (\overline{H} \cap X) \times \mathbb{P}'$, we obtain the following commutative diagram of varieties:

$$\begin{array}{ccccc} H & \xrightarrow{\kappa} & X \times \mathbb{P}' & \xrightarrow{f\times \mathrm{id}} & S \times \mathbb{P}' \\ {\scriptstyle i_H}\downarrow & & {\scriptstyle i_X\times\mathrm{id}}\downarrow & & {\scriptstyle i_S\times\mathrm{id}}\downarrow \\ \overline{H} & \xrightarrow{\overline{\kappa}} & \overline{X} \times \mathbb{P}' & \xrightarrow{\overline{f}\times \mathrm{id}} & \overline{S} \times \mathbb{P}'. \end{array} \tag{3.3.1.1}$$

In this diagram, all vertical maps are open immersions, whereas the horizontal maps $\kappa$ and $\overline{\kappa}$ are closed immersions. Let $\overline{p}\colon \overline{X} \times \mathbb{P}' \to \overline{X}$ and $p\colon X \times \mathbb{P}' \to X$ be the projections, and set $G = p^*F$ for the given perverse sheaf $F$ on $X$. The composite morphism

$$\overline{p} \circ \overline{\kappa}\colon \overline{H} \to \overline{X} \times \mathbb{P}' \to \overline{X}$$

is a projective bundle, in particular is smooth. It follows from the smooth base change theorem, which we recalled in 2.1.7, that the canonical morphism

$$\overline{\kappa}^*(i_X \times \mathrm{id})_*G = (\overline{p} \circ \overline{\kappa})^* i_{X*}F \to i_{H*}(\kappa \circ p)^*F = i_{H*}\kappa^*G \tag{3.3.1.2}$$

is an isomorphism. Pick a point of $\mathbb{P}'$ corresponding to a hyperplane $L \subseteq \mathbb{P}$. The fibre of the diagram (3.3.1.1) over this point is the diagram

$$\begin{array}{ccccc} H_L & \xrightarrow{\kappa_L} & X & \xrightarrow{f} & S \\ {\scriptstyle i_{H_L}}\downarrow & & {\scriptstyle i_X}\downarrow & & {\scriptstyle i_S}\downarrow \\ \overline{H}_L & \xrightarrow{\overline{\kappa}_L} & \overline{X} & \xrightarrow{\overline{f}\times \mathrm{id}} & \overline{S}, \end{array} \tag{3.3.1.3}$$

where $\overline{H}_L = \overline{X} \cap L$ and $H_L = X \cap L$ are the hyperplane sections given by $L$. By the smooth base change theorem, there exists a Zariski dense open subvariety of $\mathbb{P}'$ such that, for all points $L$ in this open subvariety, the base change morphisms

$$\big(\overline{\kappa}^*(i_X \times \mathrm{id})_*G\big)|_{H_L} \to \overline{\kappa}_L^* i_{X*}F \qquad \text{and} \qquad \big(i_{H*}\kappa^*G\big)|_{H_L} \to i_{H_L*}\kappa_L^*F \tag{3.3.1.4}$$

are isomorphisms. Indeed, any point $L$ around which the map $H(\mathbb{C}) \to \mathbb{P}'(\mathbb{C})$ is smooth, or just a topological fibration will do. Fix now an $L$ such that the base change morphisms (3.3.1.4) are isomorphisms and such that the hyperplane section $H_L \subset X$ has codimension $\geqslant 1$, so that its complement is dense. Since (3.3.1.2) and (3.3.1.4) are isomorphisms, the canonical morphism

$$\overline{\kappa}_L^* i_{X*} F \to i_{H_L*} \kappa_L^* F \tag{3.3.1.5}$$

obtained from (3.3.1.2) by base change is an isomorphism as well. Let $\overline{U}$ be the complement of $\overline{H}_L$ in $\overline{X}$, set $U = \overline{U} \cap X = X \setminus H_L$, and consider the diagram

$$\begin{array}{ccccc} U & \xrightarrow{j} & X & \xrightarrow{f} & S \\ \downarrow{\scriptstyle i_U} & & \downarrow{\scriptstyle i_X} & & \downarrow{\scriptstyle i_S} \\ \overline{U} & \xrightarrow{\overline{j}} & \overline{X} & \xrightarrow{\overline{f}} & \overline{S}, \end{array}$$

where $j$ and $\overline{j}$ are the inclusions. The canonical morphism

$$\overline{j}_! i_{U*}(j^* F) \to (i_X)_* j_!(j^* F) \tag{3.3.1.6}$$

is an isomorphism. This is indeed a consequence of (3.3.1.5) and the five lemma applied to the commutative diagram with exact rows

$$\begin{array}{ccccccc} \overline{j}_! i_{U*}(j^* F) & \longrightarrow & i_{X*} F & \longrightarrow & \overline{\kappa}_{L*} \overline{\kappa}_L{}^* i_{X*} F & \xrightarrow{+1} & \\ \downarrow & & \| & & \downarrow{\scriptstyle\cong} & & \\ (i_X)_* j_!(j^* F) & \longrightarrow & i_{X*} F & \longrightarrow & i_{X*} \kappa_{L*} \kappa_L{}^* F & \xrightarrow{+1} & \end{array}$$

where the righmost isomorphism is obtained by applying $\overline{\kappa}_{L*}$ to the isomorphism (3.3.1.5). Finally, we obtain from (3.3.1.6) an isomorphism

$${}^p\mathcal{H}^n(f_* j_! j^* F) = i_S^* \, {}^p\mathcal{H}^n(\overline{f}_* i_{X*} j_! j^* F) \overset{(3.3.1.6)}{\cong} i_S^* \, {}^p\mathcal{H}^n(\overline{f}_* \overline{j}_! i_{U*} j^* F) = i_S^* \, {}^p\mathcal{H}^n((\overline{f} \circ \overline{j})_! i_{U*} j^* F)$$

where we used $\overline{f}_! = \overline{f}_*$ in the last equality. The morphism $\overline{f} \circ \overline{j} \colon \overline{U} \to \overline{S}$ is affine, hence the functor $R(\overline{f} \circ \overline{j})_!$ is $t$-left exact for the perverse $t$-structure by Artin's theorem 2.1.17. It follows that the last term above vanishes for $n < 0$, thus concluding the proof. □

3.3.2 (Exponential basic lemma). — We now deduce the basic lemma for rapid decay cohomology. Below, we say that a variety has dimension $\leqslant d$ if all its irreducible components do.

Corollary 3.3.3 (Exponential basic lemma). — *Let $X$ be an affine variety of dimension $\leqslant d$, together with a regular function $f$, and let $(Y_i \to X_i \to X)_{i \in I}$ be a finite family of closed immersions. There exists a closed subvariety $Z \subseteq X$ of dimension $\leqslant d-1$ such that, for all $n \neq d$,*

$$H^n_{\mathrm{rd}}(X_i, Y_i \cup (X_i \cap Z), f) = 0.$$

Proof. Let $W \subseteq X$ be a closed subvariety of dimension $\leqslant d-1$ such that, for each $i$, the variety $X_i \setminus (W \cup Y_i)$ is either empty or smooth and equidimensional of dimension $d$. Set $W_i = X_i \cap W$.

The complex of constructible sheaves $\underline{\mathbb{Q}}_{[X_i,W_i\cup Y_i]}[d]$ is a perverse sheaf on $X$ as we have seen in Example 2.1.14. Set

$$F = \bigoplus_{i\in I} \underline{\mathbb{Q}}_{[X_i,W_i\cup Y_i]}[d]$$

and let us apply Beilinson's Theorem 3.3.1: there exists a dense open subvariety $j\colon U \hookrightarrow X$ such that ${}^p\mathcal{H}^n(Rf_*j_!j^*F) = 0$ for $n < 0$, in particular

$${}^p\mathcal{H}^n(Rf_*j_!j^*\underline{\mathbb{Q}}_{[X_i,W_i\cup Y_i]}[d]) = 0$$

for each $i$. Let $Z$ be the union of the complement of $U$ and $W$. Since $U$ is dense, $Z$ has dimension $\leqslant d-1$ and since $Z$ contains $W$, we have $j_!j^*\underline{\mathbb{Q}}_{[X_i,W_i\cup Y_i]} = \underline{\mathbb{Q}}_{[X_i,Y_i\cup(X_i\cap Z)]}$. Hence, by Corollary 3.2.3,

$$H^{n+d}_{\mathrm{rd}}(X_i, Y_i\cup(X_i\cap Z), f) \cong \psi_\infty(\Pi({}^p\mathcal{H}^n(Rf_*\underline{\mathbb{Q}}_{[X_i,Y_i\cup(X_i\cap Z)]}[d]))) = 0$$

for $n < 0$, hence $H^n_{\mathrm{rd}}(X_i, Y_i\cup(X_i\cap Z), f) = 0$ for $n < d$. On the other hand, since $X_i$ is affine of dimension $\leqslant d$, Artin's vanishing theorem shows that $H^n_{\mathrm{rd}}(X_i, Y_i\cup(X_i\cap Z), f) = 0$ for $n > d$. $\square$

## 3.4. Preliminaries on the real blow-up

The two previous descriptions of rapid decay cohomology involve passing to the limit when the real part of the function goes to infinity. We reinterpret these constructions as the cohomology of a manifold with boundary, where the boundary might not be smooth but have corners. In a sense, the limit is now taken over the ambient space itself.

3.4.1 (The real blow-up of $\mathbb{P}^1$ at infinity). — We write $\widetilde{\mathbb{P}}^1$ for the compactification of $\mathbb{C}$ by a circle at infinity, that is $\widetilde{\mathbb{P}}^1 = \mathbb{C}\sqcup S^1$. A system of open neighbourhoods of $z \in S^1 = \{w\in\mathbb{C} \mid |w| = 1\}$ is given by the sets

$$\{w\in\mathbb{C} \mid |w| > R, |\arg(w) - \arg(z)| < \varepsilon\} \sqcup \{z'\in S^1 \mid |\arg(z') - \arg(z)| < \varepsilon\}$$

for large $R$ and small $\varepsilon$, see Figure 3.4.1. For a complex number $z$ of norm 1, we will write $z\infty$ for the element of the boundary $\partial\widetilde{\mathbb{P}}^1 = S^1$ of $\widetilde{\mathbb{P}}^1$ with argument $\arg(z)$. There is a canonical map $\pi\colon \widetilde{\mathbb{P}}^1 \to \mathbb{P}^1$ sending the circle at infinity to $\infty\in\mathbb{P}^1$. We call $\pi\colon \widetilde{\mathbb{P}}^1 \to \mathbb{P}^1$ the real blow-up of $\mathbb{P}^1$ at infinity. For a real number $r$, we denote by $\widetilde{S}_r$ the union of the half plane $S_r = \{z\in\mathbb{C} \mid \mathrm{Re}(z) \geqslant r\}$ and the closed half circle at infinity $\{z\infty \in \partial\widetilde{\mathbb{P}}^1 \mid \mathrm{Re}(z) \geqslant 0\}$, as displayed in Figure 3.4.1.

3.4.2. — Let us recapitulate how the real oriented blow-up of a complex variety $X$ along a subvariety is constructed. We follow the exposition in [**Gil11**]. Let $\pi\colon L \to X$ be a complex line bundle on $X$, and let $s$ be a section of $L$. We consider the subspace $B^*_{L,s}$ of $L$ whose elements in a fibre $L_x = \pi^{-1}(x)$ are those non-zero $l \in L_x$ satisfying $r\cdot l = s(x)$ for some non-negative real

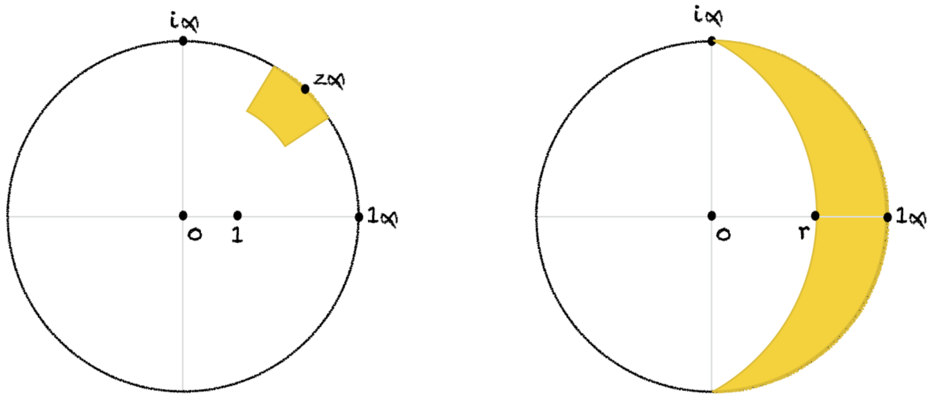



FIGURE 3.4.1. A neighbourhood of $z\infty$ (left) and the closed region $\widetilde{S}_r$ of $\widetilde{\mathbb{P}}^1$ for $r \gg 0$ (right)

number $r$. The fibre of $B^*_{L,s}$ over $x \in X$ is thus the set of positive real multiples $\mathbb{R}_{>0} \cdot s(x)$ whenever $s(x)$ is non-zero, or the set $L_x \setminus \{0\}$ in case $s(x) = 0$. The quotient topological space

$$\mathrm{Blo}_{L,s}X = B^*_{L,s}/\mathbb{R}_{>0}$$

is called the *real oriented* blow-up of $X$ along $(L, s)$. It is a closed, real semialgebraic subspace of the oriented circle bundle $S^1L = L^*/\mathbb{R}_{>0} = \mathrm{Blo}_{L,0}X$. Two sections of $L$ which differ by a nowhere vanishing function define the same real blow-up, hence it makes sense to define the real blow-up of $X$ along a Cartier divisor $D$ as

$$\mathrm{Blo}_D X = \mathrm{Blo}_{\mathcal{O}(D),s}X,$$

where $s$ is a section of $\mathcal{O}(D)$ with $D$ as zero locus. Finally, if $Z \subseteq X$ is an arbitrary closed subvariety, we define the real blow up of $X$ along $Z$ as

$$\mathrm{Blo}_Z X = \mathrm{Blo}_E(\mathrm{Bl}_Z X)$$

where $E \subseteq \mathrm{Bl}_Z X$ is the exceptional divisor in the ordinary blow-up of $X$ along $Z$. The real blow-up comes with a map $\pi\colon \mathrm{Blo}_Z X \to X$, and we call

$$\partial\mathrm{Blo}_Z X = \pi^{-1}(Z)$$

the boundary of $\mathrm{Blo}_Z X$. If $X$ is smooth and $Z$ a smooth subvariety, then $\mathrm{Blo}_Z X$ has canonically the structure of a real manifold with boundary $\partial\mathrm{Blo}_Z X$.

EXAMPLE 3.4.3. — We can view $\{\infty\} \subseteq \mathbb{P}^1$ as a Cartier divisor. The real blow-up of $\{\infty\}$ on $\mathbb{P}^1$ as constructed in 3.4.1 yields the ad-hoc construction given in 3.4.1. A bit more generally, consider the Cartier divisor $D = n \cdot [0]$ on $\mathbb{A}^1$ for some integer $n \geqslant 1$. It is described by the section $z \longmapsto z^n$ of the trivial line bundle on $\mathbb{A}^1$. The blow-up map $\pi\colon \mathrm{Blo}_D\mathbb{A}^1 \to \mathbb{A}^1$ is an isomorphism above $\mathbb{A}^1 \setminus \{0\}$, and the fibre over $\{0\}$ is the circle $S^1 = \mathbb{C}^\times/\mathbb{R}_{>0} \cong \{z \in \mathbb{C}^\times \mid |z| = 1\}$. In order to understand the topology of $\mathrm{Blo}_D\mathbb{A}^1$ we need to describe neighbourhoods of points in the boundary. By definition, $\mathrm{Blo}_D\mathbb{A}^1$ is the quotient of

$$B^*_{\mathbb{C},z^n} = \{(z, w) \in \mathbb{C} \times \mathbb{C}^\times \mid z^n w^{-1} \in \mathbb{R}_{\geqslant 0}\}$$

by the action of $\mathbb{R}_{>0}$ by multiplication on the second coordinate. Equivalently, we can and will regard $\mathrm{Blo}_D\mathbb{A}^1$ as the subspace

$$\mathrm{Blo}_D\mathbb{A}^1 = \{(z,w) \in \mathbb{C} \times S^1 \mid z^n w^{-1} \in \mathbb{R}_{\geqslant 0}\}$$

of the trivial circle bundle on $\mathbb{A}^1$. Elements of the boundary are of the form $(0, w_0)$ with $w_0$ a complex number of norm 1. A fundamental system of neighbourhoods of $(0, w_0)$ is then given by the sets

$$U = \{(z,w) \in \mathbb{C} \times S^1 \mid z^n w^{-1} \in \mathbb{R}_{\geqslant 0},\ |z| < \varepsilon,\ \arg(ww_0^{-1}) < \varepsilon\}$$

for small $\varepsilon > 0$. The intersection of $U$ with the boundary is a small arc of circle around $w_0 \in S^1$. On the other hand, the intersection of $U$ with $\mathbb{A} \setminus \{0\}$ consists of the $n$ small sectors

$$\{z \in \mathbb{C} \mid 0 < |z| < \varepsilon,\ |\arg(z^n w_0^{-1})| < \varepsilon\}$$

around the missing origin in $\mathbb{A} \setminus \{0\}$. We might thus describe the real blow-up of $\mathbb{P}^1$ at $D = n \cdot [0]$ topologically as gluing a disk to a circle via the $n$-fold covering map of the circle by itself. In particular, the topological space $\mathrm{Blo}_D\mathbb{A}^1$ does not admit the structure of a real manifold with boundary unless $n = 1$.

3.4.4. — Let $X$ be a complex variety, and let $Z_1, Z_2, \ldots, Z_m$ be closed subvarieties of $X$. We define the real oriented blow up of $X$ in centres $(Z_1, Z_2, \ldots, Z_m)$ to be

$$\mathrm{Blo}_{(Z_1,\ldots,Z_m)}X = \mathrm{Blo}_{Z_1}X \times_X \mathrm{Blo}_{Z_2}X \times_X \cdots \times_X \mathrm{Blo}_{Z_m}X,$$

where the fibre products are taken in the category of topological spaces. We will work with real blow-ups in multiple centres in the particular case where $X$ is a smooth complex variety and the $Z_i$ are the irreducible components of a normal crossings divisor. In such a situation, we may describe the real blow-up in terms of local coordinates. Suppose $X = \mathbb{A}^n = \operatorname{Spec}\mathbb{C}[x_1, \ldots, x_n]$ and let $D_p \subseteq X$ be the Cartier divisor given by $x_p = 0$. As in example 3.4.3, the real blow-up of $X$ at $(D_1, \ldots, D_m)$ can be identified with the subspace

$$\mathrm{Blo}_{(D_1,\ldots,D_m)}\mathbb{A}^n = \{(z_1, \ldots, z_n, w_1, \ldots, w_m) \in \mathbb{C}^n \times (S^1)^m \mid z_p w_p^{-1} \in \mathbb{R}_{\geqslant 0} \text{ for } 1 \leqslant p \leqslant m\}$$

of the trivial torus bundle $\mathbb{C}^n \times (S^1)^m \to \mathbb{C}^n$. The fibre over a point $(z_1, \ldots z_n) \in \mathbb{A}^n$ is a torus whose dimension is the number of zeroes in the vector $(z_1, \ldots, z_m)$. A standard neighbourhood of a point $(z_1, \ldots, z_n, w_1, \ldots, w_m)$ is given by

$$\prod_{p=1}^{m} \left\{(z,w) \in \mathbb{C} \times S^1 \;\middle|\; zw \in \mathbb{R}_{\geqslant 0}, |z - z_p| < \varepsilon, \arg(ww_p^{-1}) < \varepsilon\right\} \times \prod_{p=m+1}^{n} \left\{z \in \mathbb{C} \;\middle|\; |z - z_p| < \varepsilon\right\}$$

where in the first product, for sufficiently small $\varepsilon$, each factor is either an open disk in the case $z_p \neq 0$, or a half disk with boundary in the case $z_p = 0$. From this description, we see that in general, $\mathrm{Blo}_{(D_1,\ldots,D_m)}\mathbb{A}^n$ does not admit the structure of a real manifold with boundary, at least not in a straightforward way. However, we have seen that for a general $n$-dimensional smooth complex variety $X$ and normal crossings divisor $D_1 + \cdots + D_m$, the real blow-up $\mathrm{Blo}_{(D_1,\ldots,D_m)}X$ is locally homeomorphic to a product $[0,1)^a \times (0,1)^b$ with $a + b = 2n$, homeomorphisms depending on the choice of local coordinates. Such a beast is called a *manifold with corners*[1]. Notice that in example 3.4.3, the real oriented blow-up of $\mathbb{P}^1$ in $n \cdot [0]$ is not a manifold with corners for $n > 1$.

3.4.5 (The collar neighbourhood theorem). — The classical *collar neighbourhood theorem* for a manifold with boundary $M$ states that the boundary $\partial M$ admits a neighbourhood in $M$ which is diffeomorphic to $\partial M \times [0, 1)$, see *e.g.* [**Hir76**, §4.6]. For manifolds with corners a similar statement is true, except that one can of course not ask for a diffeomorphism.

THEOREM 3.4.6 (Collar neighbourhood theorem). — *Let $B$ be a real manifold with corners. The boundary $\partial B$ of $B$ admits an open neighbourhood which is homeomorphic to $\partial B \times [0, 1)$. In particular, if $C$ is any subset of $\partial B$, then the inclusion $B \setminus C \hookrightarrow B$ is a homotopy equivalence.*

We do not know of a reference for this proposition as it is stated. There is an ad-hoc construction of rounding corners: a manifold with corners is homeomorphic to a manifold with boundary via a homeomorphism respecting the boundaries. This procedure is described in the appendix *Arrondissement des variétés à coins* by Douady and Herault to [**BS73**]. Having rounded off the corners, one can apply the classical collar neighbourhood theorem [**Hir76**, §4.6]. Alternatively, we can avoid the rounding of corners by generalising one of the proofs of the classical collar neighbourhood theorem to manifolds with corners. Let us suppose for simplicity that the boundary $\partial B$ is compact. In a first step we construct an inward pointing vector field $F$ on $B$. Locally, on a chart $[0, 1)^a \times (0, 1)^b$ one can make an explicit choice of such a vector field, and using a partition of unity these vector fields can be glued together to a global one. Consider the associated flow $\varphi\colon \partial B \times \mathbb{R} \to B$, restricted to the boundary. By definition, this means that $\varphi(b, 0) = b$ and $\frac{\partial\varphi}{\partial t}\varphi(b, t) = F(\varphi(b, t))$. Locally, this flow exists and is unique for small times $0 \leqslant t < \varepsilon$, and since $\partial B$ is compact, we may assume that $\varphi$ is globally well defined for small times. The flow $\varphi\colon \partial B \times [0, \varepsilon) \to B$ is then locally a homeomorphism. Again using compactness of the boundary we see that we may choose a smaller $\varepsilon$ is necessary so that $\varphi$ is injective and a homeomorphism onto its image, and thus yields the collar neighbourhood.

## 3.5. Rapid decay cohomology as the cohomology of a real blow-up

3.5.1. — Let $\overline{X}$ be a smooth and compact complex manifold. Let $f : X \to \mathbb{P}^1$ be a meromorphic function with divisor of poles $P = f^{-1}(\infty)$, and let $H$ be another divisor on $X$. Suppose that $D = P + H$ is a normal crossing divisor and set $X := \overline{X} \setminus D$. Denote by $\pi\colon B \to \overline{X}$ the real blow-up of $\overline{X}$ in the components of $D$. The function $f$ lifts uniquely to a function $f_B\colon B \to \widetilde{\mathbb{P}}^1$

[1]It appears that several, inequivalent definitions of manifolds with corners are in use. Our example fits all of them as far as we know. We use Douady's definition in [**Dou61**, §I.4], which seems to be the one most adapted to our situation.

such that the diagram

$$\begin{array}{ccc} B & \xrightarrow{f_B} & \widetilde{\mathbb{P}}^1 \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ \overline{X} & \xrightarrow{f} & \mathbb{P}^1 \end{array}$$

commutes. The local description of $f_B$ is as follows: around a given point $0 \in \overline{X}$, we can choose local coordinate functions $x_1, \ldots, x_n$ such that the function $f$ is

$$f(x_1, \ldots, x_n) = \frac{f_1(x_1, \ldots, x_n)}{x_1^{e_1} x_2^{e_2} \cdots x_m^{e_m}}$$

for a function $f_1$ such that.... exponents $e_p \geqslant 0$, and the divisor $D$ is given by $x_1 x_2 \cdots x_m = 0$, with $0 \leqslant m \leqslant n$. As described in 3.4.4, the map $\pi\colon B \to X$ is given, locally around $0 \in X$, by the projection of

$$\{(x_1, \ldots, x_n, w_1, \ldots, w_m) \in \mathbb{C}^n \times (S^1)^m \mid x_p w_p^{-1} \in \mathbb{R}_{\geqslant 0} \text{ for } 1 \leqslant p \leqslant m\}$$

onto the coordinates $(x_1, \ldots, x_n)$. The map $f_B\colon B \to \widetilde{\mathbb{P}}^1$ is given by

$$f_B(x_1, \ldots, x_n, w_1, \ldots, w_m) = \begin{cases} (x_1^{e_1} x_2^{e_2} \cdots x_m^{e_m})^{-1} & \text{if } x_1^{e_1} x_2^{e_2} \cdots x_m^{e_m} \neq 0, \\ (w_1^{e_1} w_2^{e_2} \cdots w_m^{e_m})^{-1} & \text{if } x_1^{e_1} x_2^{e_2} \cdots x_m^{e_m} = 0, \end{cases}$$

so that $f_B$ maps the divisor of poles $P$ to the circle at infinity in $\widetilde{\mathbb{P}}^1$. Outside $P$, where $f$ is regular, $f_B$ is the composite of $f\colon \overline{X} \setminus P \to \mathbb{C}$ with the inclusion $\mathbb{C} \subseteq \widetilde{\mathbb{P}}^1$. Let us set $\partial^+ B := f_B^{-1}(\partial^+ \widetilde{\mathbb{P}}^1)$, where $\partial^+ \widetilde{\mathbb{P}}^1$ is the set of those $b \in \partial \widetilde{\mathbb{P}}^1$ with $\mathrm{Re}(b) \geqslant 0$. Let us denote by $B^\circ \subseteq B$ the subset

$$B^\circ := B \setminus \{b \in \partial B \mid \pi(b) \notin P \text{ or } \mathrm{Re}(f_B(b)) \leqslant 0\}$$

of $B$. This is also a manifold with boundary, the boundary $\partial B^\circ$ is the set of those $b \in \partial B$ for which $\pi(b)$ is on a pole of $f$, and $f_B(b)$ has strictly positive real part.

Proposition 3.5.2. — *In the situation of 3.5.1, the linear maps*

$$H^n(X, f^{-1}(S_r)) \xleftarrow{\cong} H^n(B, f_B^{-1}(\widetilde{S}_r)) \xrightarrow{\cong} H^n(B, \partial^+ B) \xleftarrow{\cong} H^n(B^\circ, \partial B^\circ)$$

*induced by inclusions of pairs of topological spaces are isomorphisms for large enough real $r \gg 0$.*

Proof. The inclusions $X = \overline{X} \setminus D \subseteq B$ and $f^{-1}(S_r) \subseteq f_B^{-1}(\widetilde{S}_r)$ are homotopy equivalences by the Collar Neighbourhood Theorem, whence the isomorphism on the left, with no constraint on $r$. That the right hand side morphism is an isomorphism for large $r$ is essentially a consequence of the proper base change theorem. Let $[0, \infty]$ be the real half line completed by a point at infinity. Consider the subspace

$$C := \{(b, r) \in B \times [0, \infty] \mid f_B(b) \in \widetilde{S}_r\}$$

of $B \times [0, \infty]$. The projection map $\mathrm{pr} : B \times [0, \infty] \to [0, \infty]$ is proper because $\overline{X}$, and hence $B$, is compact. Therefore, by the proper base change theorem, the canonical morphism

$$R^n \mathrm{pr}_*(\underline{\mathbb{Q}}_{[B \times [0,\infty], C]})_\infty \xrightarrow{\cong} H^n(B, \partial^+ B)$$

is an isomorphism. On the left hand side stands the stalk at $\infty$ of the sheaf on $[0,\infty]$ associated with the presheaf $U \longmapsto H^n(B\times U, (B\times U)\cap C)$. The sets $[r,\infty]$ for $0 \leqslant r < \infty$ form a fundamental system of closed neighbourhoods of $\infty \in [0,\infty]$, hence this stalk is by definition the colimit

$$\operatorname*{colim}_{r<\infty} H^n(B\times[r,\infty], (B\times[r,\infty])\cap C)$$

as $r$ goes to $\infty$. The pair $(B, f_B^{-1}(\widetilde{S}_r))$ is a deformation retract of $H^n(B\times[r,\infty], (B\times[r,\infty])\cap C)$, hence this colimit is the same as

$$\operatorname*{colim}_{r<\infty} H^n(B, f_B^{-1}(\widetilde{S}_r))$$

which eventually stabilises. Finally, the inclusion of pairs $(B^\circ, \partial B^\circ) \to (B, \partial^+ B)$ is a homotopy equivalence, again by the Collar Neighbourhood Theorem. □

From Proposition 3.5.2, we immediately derive:

Corollary 3.5.3. — *In the situation of 3.5.1, there is a canonical isomorphism of vector spaces*

$$H^n_{\mathrm{rd}}(X,f) \cong H^n(B,\partial^+B) \cong H^n(B^\circ, \partial B^\circ)$$

3.5.4. — Let us again consider the situation of 3.5.1: Proposition 3.5.2 states that the rapid decay cohomology of $(X\setminus D, f)$ is canonically isomorphic to the cohomology of the pair $(B,\partial^+B)$. Let us denote by $\kappa\colon (B\backslash\partial^+B) \to B$ the inclusion, and write $\mathbb{Q}_{[B,\partial^+B]} = \kappa_!\kappa^*\underline{\mathbb{Q}}_B$. The cohomology of the pair $(B,\partial^+B)$ is the cohomology of $B$ with coefficients in the sheaf $\mathbb{Q}_{[B,\partial^+B]}$, hence a canonical isomorphism

$$H^n(X\setminus D, f) \cong H^n(B, \mathbb{Q}_{[B,\partial^+B]}) \cong H^n(X, R\pi_*\mathbb{Q}_{[B,\partial^+B]})$$

where $\pi\colon B \to X$ is the blow-up map. Let us examine the object $R\pi_*\mathbb{Q}_{[B,\partial^+B]}$ in the derived category of sheaves on $X$.

3.5.5. — Here is a topological preparation which will eventually help us to get a better grasp on $R\pi_*\underline{\mathbb{Q}}_{[B,\partial^+B]}$. Let $m \geqslant 1$ be an integer, and let $T \subseteq \mathbb{R}^m/\mathbb{Z}^m$ be the subset defined by

$$T = \{(x_1,\ldots,x_m) \mid de_1x_1 + \cdots + de_mx_m \equiv 0 \bmod 1\}$$

for some integer $d \geqslant 1$ and primitive vector $e = (e_1,\ldots,e_m) \in \mathbb{Z}^m$. Here, primitive means not divisible in $\mathbb{Z}^m$ by an integer $\geqslant 2$, and in particular nonzero. We propose ourselves to find an explicit description of the homology groups $H_p(\mathbb{R}^m/\mathbb{Z}^m, T)$. The subspace $T$ has $d$ connected components, namely, $T$ is the disjoint union of the subtorus

$$T_0 = \{(x_1,\ldots,x_m) \mid e_1x_1 + \cdots + e_mx_m \equiv 0 \bmod 1\}$$

and its translates $T_k := T_0 + (0,\ldots,0,\frac{k}{d})$. The pair of spaces $(\mathbb{R}^m/\mathbb{Z}^m, T)$ is homeomorphic to the product of $T_0$, which is a torus of dimension $m-1$, and the circle $\mathbb{R}/\mathbb{Z}$ marked in the $d$ points $\frac{1}{d}\mathbb{Z}/\mathbb{Z}$. The pair $(\mathbb{R}/\mathbb{Z}, \frac{1}{d}\mathbb{Z}/\mathbb{Z})$ has homology in degree 1 only, and therefore the cross-product morphism

$$H_{p-1}(T_0) \times H_1(\mathbb{R}/\mathbb{Z}, \tfrac{1}{d}\mathbb{Z}/\mathbb{Z}) \to H_p(\mathbb{R}^m/\mathbb{Z}^m, T)$$

is an isomorphism. Fix a $\mathbb{Z}$-basis $a_1, \ldots, a_{m-1}$ of the orthogonal complement of $e$ in $\mathbb{Z}^m$. For any non-decreasing injective map $f\colon \{1,2,\ldots,p-1\} \to \{1,2,\ldots,m-1\}$, and any $k \in \{0,1,\ldots,d-1\}$, the continuous map

$$c_{f,k}\colon [0,1]^p \to \mathbb{R}^m/\mathbb{Z}^m \qquad\qquad c_{f,k}(t_1,\ldots,t_p) = \tfrac{k}{d}t_p + \sum_{i=1}^{p-1} a_{f(i)}t_i$$

represents an element in $H_p(\mathbb{R}^m/\mathbb{Z}^m, T)$, once we decompose the cube $[0,1]^p$ appropriately into a sum of simplicies. Together, these elements form a basis of $H_p(\mathbb{R}^m/\mathbb{Z}^m, T)$. The dimension of $H_p(\mathbb{R}^m/\mathbb{Z}^m, T)$ is $d\binom{m-1}{p-1}$.

Proposition 3.5.6. — *Set $n = \dim X$. The homology sheaves $R^p\pi_*\mathbb{Q}_{[B,\partial^+B]}$ are constructible and vanish for $p > n$. Therefore, $R\pi_*\mathbb{Q}_{[B,\partial^+B]}$ is an object of the derived category of constructible sheaves on $X$. The sheaf*

$$R\pi_*\mathbb{Q}_{[B,\partial^+B]}[n]$$

*is a perverse sheaf on $X$. Its Verdier dual is the sheaf $R\pi_*\mathbb{Q}_{[B,\partial^0B]}[n]$ where $\partial^0B \subseteq \partial B$ is the closure of the subset $\partial B \setminus \partial^+B$ of the boundary.*

Proof. The blow-up map $\pi\colon B \to X$ is proper, hence for every $x \in X$ the stalk at $x$ of the sheaf $R^p\pi_*\mathbb{Q}_{[B,\partial^+B]}$ is identical to $H^p(\pi^{-1}(x), \pi^{-1}(x) \cap \partial^+B)$. This shows that $R^p\pi_*\mathbb{Q}_{[B,\partial^+B]}$ is constructible with respect to the stratification given by intersections of the components of $D$. Precisely, if we denote by $D^{(m)} \subseteq X$ the smooth subvariety of codimension $m$ given by the union of all intersections of $m$ distinct components of $D$, then

$$\varnothing \subseteq D^{(n)} \subseteq D^{(n-1)} \subseteq \cdots \subseteq D^{(2)} \subseteq D \subseteq X$$

is a stratification for $R^p\pi_*\mathbb{Q}_{[B,\partial^+B]}$, for all $p$. The fibre $\pi^{-1}(x)$ is a real torus of real dimension $m$, where $m \leqslant n$ is the number of components of $D$ meeting at $x$, and $\pi^{-1}(x) \cap \partial^+B$ is either empty or a finite union of real tori of dimension $m-1$. In view of 3.5.5 we can be more explicit: If $x$ is in the intersection of components $D_1, D_2, \ldots, D_m$ of $D$, and $f$ has a pole of order $e_i \geqslant 0$ on $D_i$, then the stalk of $R^p\pi_*\mathbb{Q}_{(B,\partial^+B)}$ at $x$ has dimension $\gcd(e_1,\ldots,e_m)\binom{m-1}{p-1}$ if $f$ has a pole at $x$, *i.e.* at least one of the $e_i$ is nonzero, and dimension $\binom{m}{p}$ if $f$ is regular at $x$. In either case, $H^p(\pi^{-1}(x), \pi^{-1}(x) \cap \partial^+B) = 0$ if $p > m$, hence the inclusion

$$\operatorname{supp}\left(R^p\pi_*\mathbb{Q}_{[B,\partial^+B]}\right) \;\subseteq\; D^{(p)} \tag{3.5.6.1}$$

holds. Next, we compute the dual of $R^p\pi_*\mathbb{Q}_{[B,\partial^+B]}$. Let us denote by $\omega_{B/X} = \pi^!\mathbb{Q}_X$ the relative dualising sheaf of $\pi\colon B \to X$. Local Verdier duality reads

$$R\mathcal{H}om(R\pi_*\mathbb{Q}_{[B,\partial^+B]}, \mathbb{Q}) \cong R\pi_*R\mathcal{H}om(\mathbb{Q}_{[B,\partial^+B]}, \omega_{B/X})$$

hence it suffices to produce a canonical isomorphism

$$R\mathcal{H}om(\mathbb{Q}_{[B,\partial^+B]}, \omega_{B/X}) = \mathbb{Q}_{[B,\partial^0B]}$$

in the derived category of sheaves on $B$. Let us name the inclusions

$$\alpha\colon B \setminus \partial B \to B \qquad\qquad \kappa\colon B \setminus \partial^+B \to B \qquad\qquad \lambda\colon B \setminus \partial^0B \to B$$

so that $\mathbb{Q}_{(B,\partial^+B)} = \kappa_!\kappa^*\mathbb{Q}_B$ and $\mathbb{Q}_{(B,\partial^0 B)} = \lambda_!\lambda^*\mathbb{Q}_B$. Since $X$ is smooth of real dimension $2n$, the dualising sheaf on $X$ is $\omega_X = \mathbb{Q}_X[2n]$, hence we can compute the relative dualising sheaf $\omega_{B/X}$ as $\omega_B[-2n]$. We find $\omega_{B/X} = \alpha_!\alpha^*\mathbb{Q}_B$, as we would for any $C^0$-manifold with boundary. Notice that for any sheaf $F$ on $B$ there is a natural isomorphism $\mathcal{H}om(\kappa_!\kappa^*\mathbb{Q}_B, F) = \kappa_*\kappa^*F$, hence we find in particular an isomorphism

$$R\mathcal{H}om(\kappa_!\kappa^*\mathbb{Q}_B, \omega_{B/X}) = R\kappa_*\kappa^*\alpha_!\alpha^*\mathbb{Q}_B = \kappa_*\kappa^*\alpha_!\alpha^*\mathbb{Q}_B$$

in the derived category of sheaves on $B$. The functor $\kappa_*$ is exact, hence the equality on the right. Inspecting sections, we find $\kappa_*\kappa^*\alpha_!\alpha^*\mathbb{Q}_B = \lambda_!\lambda^*\mathbb{Q}_B$ as we wanted to show. For any $p \geqslant 0$, the direct image $R^p\pi_*\mathbb{Q}_{(B,\partial^0 B)}$ is a constructible sheaf on $X$, and since $\pi$ is proper, we can compute its stalks using proper base change: the stalk at $x$ is isomorphic to $H^p(\pi^{-1}(x), \pi^{-1}(x) \cap \partial^0 B)$. The fibre $\pi^{-1}(x)$ is still a real torus of real dimension equal to the number $m$ of components of $D$ meeting at $x$, and $\pi^{-1}(x) \cap \partial^0 B$ is either all of $\pi^{-1}(x)$ in case $f$ is regular on one of the components of $D$ meeting at $x$, or else, a finite union of real tori of dimension $m-1$. In either case, $H^p(\pi^{-1}(x), \pi^{-1}(x) \cap \partial^0 B) = 0$ if $p > m$, hence the inclusion

$$\operatorname{supp}\left(R^p\pi_*\mathbb{Q}_{[B,\partial^0 B]}\right) \;\subseteq\; D^{(p)} \tag{3.5.6.2}$$

holds. Together, the inclusions (3.5.6.1) and (3.5.6.2) show that $\mathbb{Q}_{[B,\partial^+ B]}[n]$ is perverse. □

3.5.7 (Good compactifications). — In 3.5.1 and Proposition 3.5.2 we started with a smooth and compact complex manifold $X$ and a function $X \to \mathbb{P}^1$, restricting to $X \setminus D \to \mathbb{A}^1$ for some normal crossings divisor $D$. In practice, we usually start with a smooth variety $X$ and a function $f\colon X \to \mathbb{A}^1$, and seek to compactify $X$ by a normal crossings divisor in such a way that $f$ extends to a function with values in $\mathbb{P}^1$ on the compactification.

Definition 3.5.8. — Let $X$ be a smooth variety over $k$, let $Y \subseteq X$ be a normal crossing divisor, and let $f\colon X \to \mathbb{A}^1$ be a regular function. A *good compactification* of $(X, Y, f)$ is a triple $(\overline{X}, \overline{Y}, \overline{f})$ consisting of a smooth projective variety $\overline{X}$ over $k$ containing $X$ as the complement of a normal crossing divisor $D$, a divisor $\overline{Y} \subseteq X$ such that $Y = \overline{Y} \cap X$ and that $\overline{Y} + D$ has normal crossings, and a morphism $\bar{f}\colon \overline{X} \to \mathbb{P}^1$ extending $f$.

3.5.9. — The situation of Definition 3.5.8 one has a commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & \overline{X} & \longleftarrow & D \\ {\scriptstyle f}\downarrow & & {\scriptstyle \overline{f}}\downarrow & & \downarrow \\ \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1 & \leftarrow & \{\infty\} \end{array}$$

where all horizontal maps are inclusions. A good compactification of $(X, Y, f)$ always exists. Indeed, let $\overline{X}_0$ be any smooth compactification of $X$ by a normal crossing divisor $D$, such that also $\overline{Y}_0 + D$ is a normal crossings divisor, where $\overline{Y}_0$ is the closure of $Y$ in $\overline{X}_0$. Such a compactification can be "found" using resolutions of singularities. The function $f$ extends to a rational map $\overline{X}_0 \dashrightarrow \mathbb{P}^1$. By resolution of indeterminacies, there exists a finite tower of blow-ups $\overline{X} \to \overline{X}_{m-1} \to \cdots \to \overline{X}_0$

at smooth centers of $D$ such that $f$ extends to a morphism $\bar{f}\colon \overline{X} \to \mathbb{P}^1$. Define $\overline{Y}$ to be the strict transform of $\overline{Y}_0$ in $\overline{X}$.

Example 3.5.10. — Let $X = \mathbb{A}^2 = \operatorname{Spec} \mathbb{Q}[x, y]$, together with the function $f = x^2 + y^2$. We start with the compactification $\overline{X}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and the rational map

$$\begin{aligned} \mathbb{P}^1 \times \mathbb{P}^1 &\dashrightarrow \mathbb{P}^1 \\ [x\colon a], [y\colon b] &\mapsto [b^2x^2 + a^2y^2\colon a^2b^2], \end{aligned}$$

whose only indeterminacy is $(\infty, \infty)$. Let $\overline{X}$ be the blow-up of this point, *i.e.* the closed subvariety of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ given by the equation $av = bu$, where $[u\colon v]$ are the coordinates of the last $\mathbb{P}^1$. Then $f$ extends to the morphism

$$\begin{aligned} \overline{X} &\xrightarrow{\bar{f}} \mathbb{P}^1 \\ [x\colon a], [y\colon b], [u\colon v] &\longmapsto [v^2x^2 + u^2y^2\colon u^2v^2]. \end{aligned}$$

The pole divisor has irreducible components $P_1 = \infty \times \mathbb{P}^1 \times [0\colon 1]$ and $P_2 = \mathbb{P}^1 \times \infty \times [1\colon 0]$, and each of them appears with multiplicity two. The horizontal component is the exceptional divisor.

Corollary 3.5.11. — *Let $X$ be a smooth complex algebraic variety with potential $f\colon X \to \mathbb{A}^1$ and let $Y \subseteq X$ be a normal crossings divisor. Let $(\overline{X}, \overline{Y}, \overline{f})$ be a good compactification of $(X, Y, f)$. Let $\pi\colon B \to \overline{X}$ be the real blow-up of $\overline{X}$ along the components of $D = \overline{X} \setminus X$, let $B_Y \subseteq B$ be the real blow-up of $Y$ along the components of $Y \cap D$, and let $\overline{f}_B\colon B \to \widetilde{\mathbb{P}}^1$ be the lift of $\overline{f}$ to $B$. There is a canonical isomorphism*

$$H^n_{\mathrm{rd}}(X, Y, f) \cong H^n(B, B_Y \cup \partial^+ B).$$

Proof. If $Y$ is empty, this is the statement of Corollary 3.5.3. If $Y$ has only one (smooth) irreducible component, there is a commutative diagram

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H^{n-1}_{\mathrm{rd}}(Y, f) & \longrightarrow & H^n_{\mathrm{rd}}(X, Y, f) & \longrightarrow & H^n_{\mathrm{rd}}(X, f) & \longrightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ \cdot \longrightarrow & H^n(B_Y, \partial^+ B_Y) & \longrightarrow & H^n(B, B_Y \cup \partial^+ B) & \longrightarrow & H^n(B, \partial^+ B) & \longrightarrow \cdots \end{array}$$

we can also apply Proposition 3.5.2 to $Y$ and deduce the statement of the corollary by dévissage. The general case is by induction on the number of irreducible components of $Y$. □

## 3.6. The Künneth formula

The classical Künneth formula relates the singular cohomology of reasonable topological spaces $X_1$ and $X_2$ to the cohomology of the product space $X_1 \times X_2$. In the case of rational coefficients, or indeed coefficients in any field, the Künneth formula simply states that the map

$$H^*(X_1, \mathbb{Q}) \otimes H^*(X_2, \mathbb{Q}) \longrightarrow H^*(X_1 \times X_2, \mathbb{Q})$$

induced by the cup-product is an isomorphism of graded vector spaces. This works equally well for pairs of spaces: given closed subspaces $Y_1 \subseteq X_1$ and $Y_2 \subseteq X_2$, the cup-product induces an isomorphism of graded vector spaces

$$H^*(X_1, Y_1; \mathbb{Q}) \otimes H^*(X_2, Y_1; \mathbb{Q}) \longrightarrow H^*(X_1 \times X_2, (X_1 \times Y_2) \cup (Y_1 \times X_2); \mathbb{Q}).$$

In this section, we introduce the cup-product for rapid decay cohomology and establish a Künneth formula in this context.

Definition 3.6.1. — Given sets (schemes, topological spaces, ...) $X_1$ and $X_2$, a commutative group (scheme, ...) $C$ and maps $f_1\colon X_1 \to C$ and $f_2\colon X_2 \to C$, the *Thom-Sebastiani sum* $f_1 \boxplus f_2$ is the map $X_1 \times X_2 \to C$ defined by the formula

$$(f_1 \boxplus f_2)(x_1, x_2) = f_1(x_1) + f_2(x_2).$$

3.6.2 (Elementary construction of the cup-product). — Let $(X_1, Y_1)$ and $(X_2, Y_2)$ be pairs consisting of a complex variety and a closed subvariety, and let $f_1\colon X_1 \to \mathbb{A}^1$ and $f_2\colon X_2 \to \mathbb{A}^1$ be regular functions. The cup product

$$H^i(X_1, Y_1, f_1) \otimes H^j(X_2, Y_2, f_2) \longrightarrow H^{i+j}(X_1 \times X_2, (Y_1 \times X_2) \cup (X_1 \times Y_2), f_1 \boxplus f_2) \tag{3.6.2.1}$$

is the linear map obtained, by passing to the limit $r \to +\infty$, from the composition

$$\begin{aligned} &H^i(X_1, Y_1 \cup f_1^{-1}(S_r)) \otimes H^j(X_2, Y_2 \cup f_2^{-1}(S_r)) \\ &\qquad \longrightarrow H^{i+j}(X_1 \times X_2, (Y_1 \times X_2) \cup (X_1 \times Y_2) \cup (f_1^{-1}(S_r) \times X_2) \cup (X_1 \times f_2^{-1}(S_r))) \\ &\qquad \longrightarrow H^{i+j}(X_1 \times X_2, (Y_1 \times X_2) \cup (X_1 \times Y_2) \cup (f_1 \boxplus f_2)^{-1}(S_{2r})), \end{aligned}$$

where the first map is the usual cup product of relative cohomology, and the second one is induced by the inclusion of closed subsets $(f_1 \boxplus f_2)^{-1}(S_{2r}) \subset (f_1^{-1}(S_r) \times X_2) \cup (X_1 \times f_2^{-1}(S_r))$.

Proposition 3.6.3 (Künneth formula). — *Let $(X_1, f_1)$ and $(X_2, f_2)$ be complex varieties with potentials and let $Y_1 \subseteq X_1$ and $Y_2 \subseteq X_2$ be closed varieties. The cup product* (3.6.2.1) *induces an isomorphism of graded vector spaces*

$$H^*(X_1, Y_1, f_1) \otimes H^*(X_2, Y_2, f_2) \xrightarrow{\cong} H^*(X_1 \times X_2, (Y_1 \times X_2) \cup (X_1 \times Y_2), f_1 \boxplus f_2).$$

Proof. The Künneth formula for relative topological spaces yields an isomorphism of graded vector spaces

$$H^*(X_1, f_1^{-1}(S_r)) \otimes H^*(X_2, f_2^{-1}(S_r)) \xrightarrow{\cong} H^*(X_1 \times X_2, (f_1^{-1}(S_r) \times X_2) \cup (X_1 \times f_2^{-1}(S_r)))$$

induced by cup products. To ease the notation, we left out $Y_1$ and $Y_2$ from the notation. We need to show that the linear map

$$H^n((f_1^{-1}(S_r) \times X_2) \cup (X_1 \times f_2^{-1}(S_r))) \to H^n((f_1 \boxplus f_2)^{-1}(S_{2r})) \tag{3.6.3.1}$$

induced by the inclusion $(f_1 \boxplus f_2)^{-1}(S_{2r}) \subset (f_1^{-1}(S_r) \times X_2) \cup (X_1 \times f_2^{-1}(S_r))$ is an isomorphism for sufficiently large real $r$. In terms of constructible sheaves, this amounts to the following: let $F_i$ be a constructible sheaf on $X_i$, say for example $F_i = \underline{\mathbb{Q}}_{[X_i,Y_i]}$, and consider the open sets

$$U := \{(z_1, z_2) = \mathrm{Re}(z_1) \geqslant r \text{ or } \mathrm{Re}(z_1) \geqslant r \qquad \text{and} \qquad V := \{(z_1, z_2) \mid \mathrm{Re}(z_1 + z_2) \geqslant 2r\}.$$

We must show that the map

$$H^n(U, (Rf_{1*}F_1 \boxtimes Rf_{1*}F_2)|_U) \to H^n(V, (Rf_{1*}F_1 \boxtimes Rf_{1*}F_2)|_V)$$

induced by the inclusion $V \subseteq U$ is an isomorphism. The homology sheaves of $Rf_{1*}F_1 \boxtimes Rf_{1*}F_2$ are constructible with respect to a stratification consisting of finitely many horizontal and vertical lines. Let $G$ be any such constructible sheaf, that is, $G$ is a sheaf on $\mathbb{C}^2$ constructible with respect to the stratification given by lines $\mathbb{C} \times \{s\}$ or $\{s\} \times \mathbb{C}$ and their intersection points, where $s$ belongs to a finite set of complex numbers $S$. Fix a real $r$ such that $r > \mathrm{Re}(s)$ for all $s \in S$, and let us show that the inclusion $V \subseteq U$ induces an isomorphism $H^n(V, G|_V) \cong H^n(U, G|_U)$. To this end, define

$$B := \{(z_1, z_2, t) \in \mathbb{C}^2 \times [0,1] \mid \mathrm{Re}(z_1 + tz_2) \geqslant r + tr \text{ or } \mathrm{Re}(tz_1 + z_2) \geqslant r + tr\}$$

and consider the sheaf $G_B = (\mathrm{pr}^*G)|_B$ on $B$. The projection $p : B \to [0,1]$ is a topological fibre bundle, its fibre over 0 is $U$ and its fibre over 1 is $V$. The sheaf $G_B$ is constructible with respect to a stratification of $B$ by subvarieties, each of which also is a fibre bundle over $[0,1]$, hence the sheaf $R^n p_* G_B$ is a local system on $[0,1]$. Parallel transport from the fibre over 0 to the fibre over 1 is the isomorphism we sought. □

3.6.4. — Here is an illustration in the real plane of the various sets considered in the proof of the Künneth formula. In this picture the horizontal and vertical lines represent the stratification

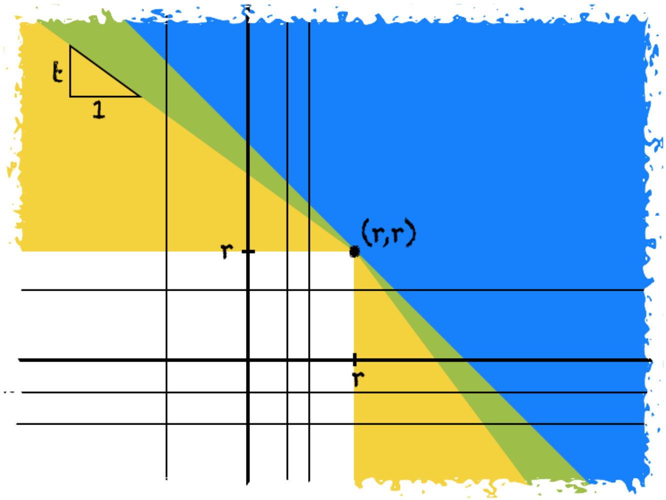



FIGURE 3.6.2. The sets $V \subseteq p^{-1}(t) \subseteq U$

for $G$, so $G$ is a local system outside these lines, and also when restricted to each of the lines except at the intersection points. The whole colored region is $U$, and the blue region is $V$. The green and blue parts together form $p^{-1}(t)$.

3.6.5. — We defined the cup product for rapid decay cohomology in 3.6.2 in elementary terms. We can also give a construction in terms of sheaf cohomology.

## 3.7. Rapid decay cohomology with support

In this section we define the rapid decay cohomology with support on some closed subvariety. As one is accustomed, this cohomology with support will fit into a long exact sequence relating it with the rapid decay cohomology of the ambient variety and the rapid decay cohomology of the open complement. We will also define a Gysin map for rapid decay cohomology, and construct the corresponding long exact Gysin sequence.

3.7.1. — Let $X$ be a variety equipped with a potential $f\colon X \to \mathbb{A}^1$, and let $Y \subseteq X$ be a closed subvariety. Let $\alpha : Z \to X$ be the inclusion of a closed subvariety with complement $\beta\colon U \to X$. We call

$$H^n_{\mathrm{rd},Z}(X,Y,f) := \Psi_\infty \Pi({}^p\mathcal{H}^n(Rf_*R\alpha_!\alpha^!\mathbb{Q}_{[X,Y]})) \tag{3.7.1.1}$$

the rapid decay cohomology in degree $n$ of $(X,Y,f)$ with support on $Z$. There is an exact triangle

$$R\alpha_!\alpha^!\mathbb{Q}_{[X,Y]} \to \mathbb{Q}_{[X,Y]} \to R\beta_*\beta^*\mathbb{Q}_{[X,Y]} \to R\alpha_!\alpha^!\mathbb{Q}_{[X,Y]}[1]$$

in the derived category of constructible sheaves on $X$. The sheaf $\beta^*\mathbb{Q}_{[X,Y]}$ on $U$ is the same as $\mathbb{Q}_{[U,Y\cap U]}$, and the functors $\Psi_\infty$ and $\Pi$ are exact. Hence we obtain a long exact sequence

$$\cdots \to H^n_{\mathrm{rd},Z}(X,Y,f) \to H^n_{\mathrm{rd}}(X,Y,f) \to H^n_{\mathrm{rd}}(U,Y\cap U,f|_U) \to H^{n+1}_{\mathrm{rd},Z}(X,Y,f) \to \cdots \tag{3.7.1.2}$$

of vector spaces. We call the morphism $H^n_{\mathrm{rd},Z}(X,Y,f) \to H^n_{\mathrm{rd}}(X,Y,f)$ the *forget supports* map. The morphism following it is the usual restriction morphism, that is, the morphism in rapid decay cohomology induced by the inclusion $U \to X$.

3.7.2. — Let $X$ be a smooth variety, together with a regular function $f$. Let $i\colon Z \hookrightarrow X$ be a smooth closed subvariety of pure codimension $c$ with complementary immersion $j\colon U \hookrightarrow X$. Recall from 2.1.6 that $i_! = i_*$ and $i^! = i^*[-2c]$, so in particular $i_!i^!\mathbb{Q} = i_*i^*\mathbb{Q}[-2c]$. The adjunction morphism for $i_!$ sits in a triangle

$$i_*i^*\mathbb{Q}[-2c] \longrightarrow \mathbb{Q} \longrightarrow Rj_*j^*\mathbb{Q}.$$

Upon application of $Rf_*$, this triangle induces a long exact sequence of perverse sheaves

$$\cdots \longrightarrow {}^p\mathcal{H}^{n-2c}(Rf_*i_*i^*\mathbb{Q}) \longrightarrow {}^p\mathcal{H}^n(Rf_*\mathbb{Q}) \longrightarrow {}^p\mathcal{H}^n(Rf_*Rj_*j^*\mathbb{Q}) \longrightarrow \cdots$$

Taking the projector $\Pi$ and the nearby fibre at infinity we find a long exact sequence

$$\cdots \longrightarrow H^{n-2c}_{\mathrm{rd}}(Z,f_{|Z}) \longrightarrow H^n_{\mathrm{rd}}(X,f) \longrightarrow H^n_{\mathrm{rd}}(U,f_{|U}) \longrightarrow \cdots \tag{3.7.2.1}$$

of rational vector spaces which is called the *Gysin long exact sequence.* The morphism

$$H^{n-2c}_{\mathrm{rd}}(Z,f_{|Z}) \longrightarrow H^n_{\mathrm{rd}}(X,f) \tag{3.7.2.2}$$

is the Gysin map for rapid decay cohomology.

## 3.8. Poincaré–Verdier duality

The goal of this section is to construct a Poincaré–Verdier duality pairing for rapid decay cohomology. To construct a natural duality pairing such as displayed in (3.8.2.1) out of local Verdier duality is an exercise in the six functors formalism. However, since later we want to show that the resulting pairing is *motivic*, in a sense yet to be made precise, a sheaf-theoretic construction is not enough for us. We will rather construct a specific pairing by geometric means, not involving local Verdier duality. Then, we will have to check that the pairing we constructed geometrically is actually a perfect pairing, by comparing it to the sheaf-theoretic construction.

3.8.1. — To say that a finite-dimensional vector space $V$ is dual to another space $W$ usually means that there is some particular linear map $p : V \otimes W \to \mathbb{Q}$, called *pairing*. This pairing has to be *perfect*, meaning that the induced maps $V \to \mathrm{Hom}(W, \mathbb{Q})$ and $W \to \mathrm{Hom}(V, \mathbb{Q})$ are isomorphisms. Less usual, but better suited to our later needs, is the point of view that to exhibit $W$ as the dual of $V$ is to give a linear map

$$c\colon \mathbb{Q} \to W \otimes V$$

called *copairing*. Again, this copairing has to be *perfect*, that is, the induced map $\mathrm{Hom}(W, \mathbb{Q}) \to V$ sending $\varphi$ to $(\varphi \otimes \mathrm{id}_V)(c(1))$ and its companion $\mathrm{Hom}(V, \mathbb{Q}) \to W$ are both isomorphisms. Given vector spaces $V$ and $W$, there is a canonical bijection between the set of perfect pairings $p\colon V \otimes W \to \mathbb{Q}$ and the set of perfect copairings $c\colon \mathbb{Q} \to W \otimes V$, as both sets are also in canonical bijection with the set of isomorphisms between $V$ and $\mathrm{Hom}(W, \mathbb{Q})$. A pairing $p$ and a copairing $c$ correspond to each other via this bijection if the composite linear map

$$V = V \otimes \mathbb{Q} \xrightarrow{\mathrm{id}_V \otimes c} V \otimes W \otimes V \xrightarrow{p \otimes \mathrm{id}_V} \mathbb{Q} \otimes V = V \tag{3.8.1.1}$$

is the identity on $V$.

3.8.2. — Let $X$ be a smooth connected variety of dimension $d$, let $Y \subseteq X$ be a normal crossing divisor, and let $f$ be a regular function on $X$. We choose a good compactification $(\overline{X}, \overline{Y}, \overline{f})$ as in Definition 3.5.8, we let $P$ be the reduced pole divisor of $\overline{f}$ and we decompose the divisor at infinity $D = \overline{X} \setminus X$ as a sum $D = P + H$. We set

$$X' = \overline{X} \setminus (\overline{Y} \cap P), \qquad Y' = H \setminus (H \cap P)$$

and denote by $f'$ the restriction of $\overline{f}$ to $X'$. Our aim is to construct a canonical duality pairing

$$H^n_{\mathrm{rd}}(X, Y, f) \otimes H^{2d-n}_{\mathrm{rd}}(X', Y', -f') \longrightarrow \mathbb{Q}(-d). \tag{3.8.2.1}$$

Observe that in the special case where $f = 0$ and $Y$ is empty, the space $X'$ is just a smooth compactification of $X$ and $H^{2d-n}_{\mathrm{rd}}(X', Y', -f')$ is the cohomology with compact support $H^{2d-n}_c(X)$.

We want to recover from (3.8.2.1) the classical Poincaré–Verdier duality pairing. For non-empty $Y$ but $f$ still zero, the resulting pairing is sometimes called "red-green duality". What we will actually construct is not directly a pairing (3.8.2.1), but rather a copairing. Set $U = X \cap X'$ and let $\Delta_U \subseteq X \times X'$ be the diagonal embedding of $U$. We call *Poincaré–Verdier copairing* the following composite linear map.

$$\begin{array}{ccc} \mathbb{Q}(-d) = H^0(\Delta)(-d) & \xrightarrow{\text{Gysin}} & H^{2d}_{\mathrm{rd}}(X \times X', Y \times X' \cup X \times Y', f \boxplus -f') \\ & & \Big\downarrow \text{Künneth} \\ & & H^n_{\mathrm{rd}}(X, Y, f) \otimes H^{2d-n}_{\mathrm{rd}}(X', Y', -f'). \end{array} \tag{3.8.2.2}$$

We can recognise this copairing as the fibre at infinity of the similarly defined copairing for perverse cohomology.

THEOREM 3.8.3. — *The Poincaré–Verdier copairing* (3.8.2.2) *is perfect.*

3.8.4. — Let us explain how the global Verdier duality theorem can be formulated in terms of copairings. Fix an object $F$ in the derived category of constructible sheaves on a complex algebraic variety $X$. We write $\Delta : X \to X \times X$ for the inclusion of the diagonal, and $\pi$ for the map from $X$ to a point, so the dualising sheaf on $X$ is the complex $\omega = \pi^!\mathbb{Q}$. The dual of the evaluation morphism $\varepsilon : \Delta^*(F \boxtimes \mathbb{D}F) = F \otimes \mathbb{D}F \to \omega$ is a morphism $\mathbb{D}\varepsilon : \mathbb{Q} \to \Delta^!(\mathbb{D}F \boxtimes F)$. Writing $\pi_*$ as the composition of $\Delta$ and $\pi^2 = \pi \times \pi$ we obtain the sequence of morphisms

$$\begin{array}{ccccc} R\pi_*\mathbb{Q} & \xrightarrow{\mathbb{D}\varepsilon} & R\pi^2_*\Delta_*\Delta^!(\mathbb{D}F \boxtimes F) & \xrightarrow{\text{Adjunction}} & R\pi^2_*(\mathbb{D}F \boxtimes F) \\ & & & & \Big\downarrow \text{Künneth} \\ & & & & R\pi_*\mathbb{D}F \otimes R\pi_*F \end{array} \tag{3.8.4.1}$$

in the derived category of vector spaces. We have used here the fact that $\Delta$ is proper, hence $\Delta_* = \Delta_!$. Taking homology in degree 0 and projecting onto some component in the Künneth formula yields a copairing

$$\mathbb{Q} = H^0(X) \to H^{-n}(X, \mathbb{D}F) \otimes H^n(X, F)$$

which is perfect, and corresponds via the linear algebra operations outlined in 3.8.1 to the usual Verdier duality pairing $H^n(X, F) \otimes H^{-n}(X, \mathbb{D}F) \to \mathbb{Q}$. To verify this fact, which we are not going to do here, one has to check that the composite as in (3.8.1.1) of the pairing and the copairing is equal to the identity, which amounts to prove that the following diagram commutes.

$$\begin{array}{ccc}
R\pi_*F \otimes \mathbb{Q} = R\pi_*F & = & R\pi_*F = \mathbb{Q} \otimes R\pi_*F \\
\downarrow \scriptstyle{\mathrm{id}\otimes \mathrm{adj.}} & & \uparrow \scriptstyle{\mathrm{adj.}\otimes \mathrm{id}} \\
R\pi_*F \otimes R\pi_*\mathbb{Q} & & \mathbb{D}R\pi_!\omega \otimes R\pi_*F \\
\downarrow \scriptstyle{\mathrm{id}\otimes \mathbb{D}\varepsilon} & & \uparrow \scriptstyle{\varepsilon\otimes \mathrm{id}} \\
R\pi_*F \otimes R\pi_*\Delta^!(\mathbb{D}F \boxtimes F) & & \mathbb{D}R\pi_!\Delta^*(\mathbb{D}F \boxtimes F) \otimes R\pi_*F \\
\downarrow \scriptstyle{\mathrm{id}\otimes \mathrm{adj.}} & & \uparrow \scriptstyle{\mathrm{adj.}\otimes \mathrm{id}} \\
R\pi_*F \otimes R\pi^2_*(\mathbb{D}F \boxtimes F) & & \mathbb{D}R\pi^2_!(\mathbb{D}F \boxtimes F) \otimes R\pi_*F \\
\downarrow \scriptstyle{\mathrm{Kü.}} & & \uparrow \scriptstyle{\mathrm{Kü.}} \\
R\pi_*F \otimes R\pi_*\mathbb{D}F \otimes R\pi_*F & = & \mathbb{D}R\pi_!\mathbb{D}F \otimes \mathbb{D}R\pi_!F \otimes R\pi_*F
\end{array}$$

Our next task is to compare the recipe for the Poincaré–Verdier copairing (3.8.2.2) with the sheaf-theoretic description of the Verdier duality copairing (3.8.4.1).

LEMMA 3.8.5. — *Let $Z$ be a smooth complex manifold and $D, D' \subset Z$ closed subvarieties such that $D \cup D'$ is a normal crossing divisor. Consider the diagram of inclusions*

$$\begin{array}{ccc}
Z \setminus (D \cup D') & \xrightarrow{\beta} & Z \setminus D \\
\downarrow \scriptstyle{\beta'} & & \downarrow \scriptstyle{\lambda} \\
Z \setminus D' & \xrightarrow{\lambda'} & Z
\end{array}$$

*There is a canonical isomorphism*

$$R\lambda'_*\beta'_!\mathbb{Q} \cong \lambda_! R\beta_*\mathbb{Q}$$

*in the derived category of constructible sheaves on $Z$.*

PROOF. □

PROPOSITION 3.8.6. — *Let $(X, Y, f)$ and $(X', Y', f')$ be as in 3.8.2. The objects*

$$\Pi({}^p\mathcal{H}^n Rf_*\mathbb{Q}_{[X,Y]}) \qquad \text{and} \qquad \Pi({}^p\mathcal{H}^{2d-n} R(-f)_*\mathbb{Q}_{[X',Y']})$$

*of $\mathbf{Perv}_0$ are dual to each other.*

Proof. We apply Lemma 3.8.5 to $Z = \overline{X} \setminus P$, $D = H \setminus (H \cap P)$ and $D' = \overline{Y} \setminus (\overline{Y} \cap P)$. Then the diagram of inclusions becomes

$$\begin{array}{ccc} X \setminus Y = X' \setminus Y' & \xrightarrow{\ \beta\ } & X \\ \big\downarrow{\scriptstyle \beta'} & & \big\downarrow{\scriptstyle \lambda} \\ X' & \xrightarrow{\ \lambda'\ } & \overline{X} \setminus P \end{array}$$

$$\mathbb{D}(R\lambda_* \mathbb{Q}_{[X,Y]}) = \mathbb{D}(R\lambda_* \beta_! \beta^* \mathbb{Q}) = \lambda_! R\beta_* \beta^* \mathbb{Q}[2d] = R\lambda'_* \beta'_! (\beta')^* \mathbb{Q}[2d] \tag{3.8.6.1}$$

And now begins the fun:

$$\begin{aligned} \Pi({}^p\mathcal{H}^n Rf_* \mathbb{Q}_{[X,Y]})^\vee &= \Pi([-1]^* \mathbb{D}(\Pi({}^p\mathcal{H}^n Rf_* \mathbb{Q}_{[X,Y]}))) \\ &= \Pi([-1]^* \mathbb{D}({}^p\mathcal{H}^n Rf_* \mathbb{Q}_{[X,Y]})) && \text{(Lemma 2.4.8)} \\ &= \Pi({}^p\mathcal{H}^{-n} [-1]^* \mathbb{D}(Rf_* \mathbb{Q}_{[X,Y]})) && (\mathbb{D} \circ {}^p\mathcal{H}^n = {}^p\mathcal{H}^{-n} \circ \mathbb{D}) \\ &= \Pi({}^p\mathcal{H}^{-n} \mathbb{D}(R(-f)_* \mathbb{Q}_{[X,Y]})) \\ &= \Pi({}^p\mathcal{H}^{-n} \mathbb{D}(R(-\overline{f})_* R\lambda_* \mathbb{Q}_{[X,Y]})) && (f = \overline{f} \circ \lambda) \\ &= \Pi({}^p\mathcal{H}^{-n} (R(-\overline{f})_* \mathbb{D}(R\lambda_* \mathbb{Q}_{[X,Y]}))) && (\overline{f} \text{ proper}) \\ &= \Pi({}^p\mathcal{H}^{-n} (R(-\overline{f})_* R\lambda'_* \beta'_! (\beta')^* \mathbb{Q}[2d])) && (3.8.6.1) \\ &= \Pi({}^p\mathcal{H}^{2d-n} (R(-f')_* \beta'_! (\beta')^* \mathbb{Q})) && (f' = \overline{f} \circ \lambda') \\ &= \Pi({}^p\mathcal{H}^{2d-n} (R(-f')_* \mathbb{Q}_{[X',Y']})). \end{aligned}$$

This is what we wanted to show. □

Proposition 3.8.7. — *There is a non-degenerate duality pairing*

$$H^n_{\mathrm{rd}}(X, Y, f) \otimes H^{2d-n}_{\mathrm{rd}}(X', Y', -f) \longrightarrow \mathbb{Q}(-d). \tag{3.8.7.1}$$

Proof. We take the real blow-up point of view on rapid decay cohomology. Let $B$ be the real blow-up of $\overline{X}$ along the components of $D$. By Corollary 3.5.11

$$\begin{aligned} H^n_{\mathrm{rd}}(X, Y, f) &\cong H^n(\overline{X}, R\pi_* \mathbb{Q}_{[B, B_Y \cup \partial^+ B]}) \\ H^{2d-n}_{\mathrm{rd}}(X', Y', -f) &\cong H^{-n}(\overline{X}, R\pi_* \mathbb{Q}_{[B, B_{Y'} \cup \partial^- B]}[2d]) \end{aligned}$$

We compute the Verdier dual: since $\pi$ is proper, by local Verdier duality (Theorem 2.1.9), one has

$$\mathbb{D}(R\pi_* \mathbb{Q}_{[B, B_Y \cup \partial^+ B]}) = R\pi_* R\mathcal{H}om(\mathbb{Q}_{[B, B_Y \cup \partial^+ B]}, \omega_{B/X})$$

□

3.8.8 (Real blow-up point of view). —

Lemma 3.8.9. — *Let $B$ be a topological manifold with boundary, of real dimension $n$, and let $\alpha\colon B \setminus \partial B \hookrightarrow B$ be the inclusion of the complement of the boundary. The dualising sheaf $\omega_B$ on $B$ is isomorphic to $\alpha_! \alpha^* \mathbb{Q}[n]$.*

PROOF. The dualising sheaf on a general topological space is not a sheaf properly, but an object in the derived category of sheaves. We have to show that $H^{-n}(\omega_B) = \alpha_!\alpha^*\mathbb{Q}$ and $H^{-p}(\omega_B) = 0$ holds for $p \neq n$. The sheaf $H^{-p}(\omega_B)$ is the sheafification of the presheaf

$$U \longmapsto H_p(\dot{U}, \{\cdot\}, \mathbb{Q})$$

where $\dot{U}$ is the one point compactification of $U$. For opens $V \subseteq U$, the restriction morphism $H_p(\dot{U}, \{\cdot\}, \mathbb{Q}) \to H_p(\dot{V}, \{\cdot\}, \mathbb{Q})$ in this presheaf is given by the morphism in homology induced by the map $\dot{U} \to \dot{V}$ contracting $U \setminus V$ to the special point $\cdot \in \dot{V}$. A point $b \in B$ which is not in the boundary has a fundamental system of neighbourhoods $U$ which are homeomorphic to an open ball of dimension $n$. The one point compactification of such a ball is a sphere of dimension $n$. We find that $H_p(\dot{U}, \{\cdot\}, \mathbb{Q})$ is zero for $p \neq n$ and equal to $\mathbb{Q}$ for $p = n$. A point $b \in \partial B$ has a fundamental system of neighbourhoods $U$ which are homeomorphic to a half ball

$$\{x = (x_1 \ldots x_n) \in \mathbb{R}^n \mid \|x\| < 1 \text{ and } x_1 \geqslant 0\}$$

whose one point compactification is a closed ball of dimension $n$. We find that $H_p(\dot{U}, \{\cdot\}, \mathbb{Q})$ is zero for all $p$. □

LEMMA 3.8.10. — *Let $B$ a topological manifold with boundary, of real dimension $n$. Assume that the boundary $\partial B$ is the union of two closed subsets $Z_1$ and $Z_2$ such that $Z_1 \cap Z_2$ has dense complement in $\partial B$. Then the Verdier dual of $\mathbb{Q}_{[B,Z_1]}$ is $\mathbb{Q}_{[B,Z_2]}[n]$.*

PROOF. Let $\lambda_i \colon B \setminus Z_i \hookrightarrow B$ denote the inclusions. By the previous lemma:

$$\mathbb{D}(\mathbb{Q}_{[B,Z_1]}) = R\mathcal{H}om((\lambda_1)_! \lambda_1^*\mathbb{Q}, \alpha_!\alpha^*\mathbb{Q}[n]).$$

□

EXAMPLE 3.8.11. — Let us describe the Poincaré–Verdier duality pairing (3.8.2.1) in the case where $X = \mathbb{A}^1 = \operatorname{Spec} k[t]$ is the affine line, $Y \subseteq X$ is empty, and $f \in k[t]$ is a unitary polynomial of degree $d \geqslant 2$. We start with the linear dual of the copairing (3.8.2.2). This is a pairing

$$\langle -, - \rangle \colon H_1^{\mathrm{rd}}(\mathbb{A}^1, f) \otimes H_1^{\mathrm{rd}}(\mathbb{A}^1, -f) \to \mathbb{Q}(1) \tag{3.8.11.1}$$

which we seek to describe in terms of the usual explicit bases for rapid decay homology of a polynomial on the affine line. Here, $\mathbb{Q}(1)$ should be read as $\mathbb{Q}(1) = H^1(S^1) \simeq \mathbb{Q}$. The following picture shows a basis $\gamma_1, \gamma_2, \ldots$ of the rapid decay homology group $H_1^{\mathrm{rd}}(\mathbb{A}^1, f)$ in green, and superposed in red a basis $\gamma_1', \gamma_2', \ldots$ for the rapid decay homology group $H_1^{\mathrm{rd}}(\mathbb{A}^1, -f)$, here in the case of a polynomial of degree $d = 7$. Importantly, we have chosen the paths $\gamma_i$ and $\gamma_i'$ in such a way that they intersect at most once, and if so, transversally. The pairing (3.8.11.1) is defined in elementary terms as follows: Choose a sufficiently large real number $r > 0$, and an open tubular neighbourhood $N\Delta$ of the diagonal $\Delta \subseteq \mathbb{A}^2$, sufficiently thin so that $N\Delta$ and $(f \boxplus -f)^{-1}(S_r)$ do not meet. Write $U \subseteq \mathbb{A}^2$ for the complement of the diagonal, and set $S\Delta = N\Delta \cap U$. The rapid decay homology $H^2(\mathbb{A}^2, f \boxplus -f)$ contains the cross-product cycles

$$\gamma_{ij} = \gamma_i \cup \gamma_j' \colon [0,1]^2 \to \mathbb{A}^2$$

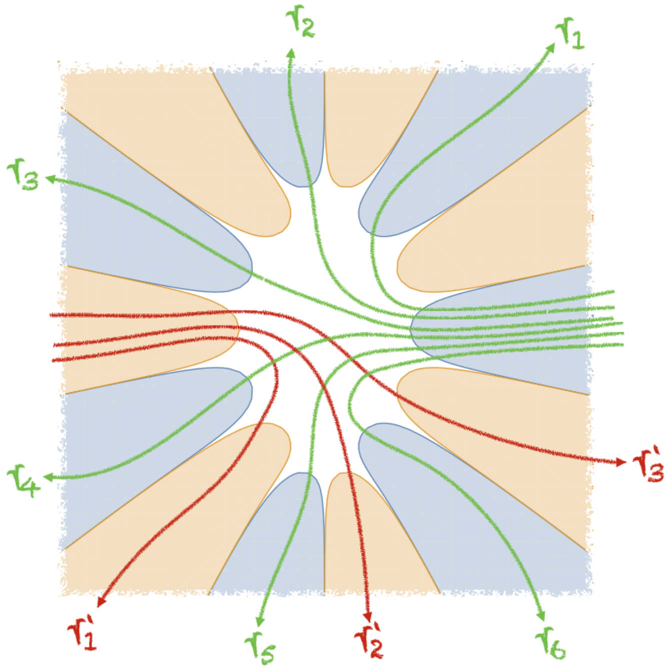



FIGURE 3.8.3. Paths $\gamma_i$ and $\gamma'_i$

defined by $\gamma_{ij}(s,t) = (\gamma_i(s), \gamma'_j(t))$. The cycles $\gamma_{ij}$ form even a basis of $H_2(\mathbb{A}^2, f \boxplus -f)$. The sought pairing 3.8.11.1 sends $\gamma_i \otimes \gamma'_j$ to the image of $\gamma_i \cup \gamma'_j$ under the connecting morphism

$$\partial\colon H_2(\mathbb{A}^2, f \boxtimes -f) \to H_1(S\Delta) \cong H_1(\Delta \times S^1) \cong \mathbb{Q}(1)$$

in the Mayer–Vietoris sequence for the covering $\mathbb{A}^2 = N\Delta \cup U$. Recall how this connecting morphism is made: Using Lebesgue's Lemma, we can write $\gamma_{ij}$ up to a boundary in the form

$$\gamma_{ij} = \alpha + \beta$$

where $\alpha$ is a cycle in $N\Delta$ and $\beta$ a cycle in $U$. Then we declare $\partial\gamma_{ij}$ to be the homology class of $d\alpha$. This already shows that if the paths $\gamma_i$ and $\gamma'_j$ do not cross, then $\langle \gamma_i, \gamma'_j \rangle = 0$, since in that case we can choose $\alpha = 0$. If $\gamma_i$ and $\gamma'_j$ do cross, then we subdivide $[0,1]$ in rectangles, sufficiently small so that the only rectangle containing the point $\gamma_{ij}^{-1}(\Delta)$ in its interior is sent to $N\Delta$. We may take for $\alpha$ the restriction of $\gamma_{ij}$ to this small rectangle, and see that the image of $\gamma_{ij}$ in $H_1(S\Delta) \cong H_1(S^1) = \mathbb{Q}(1)$ is $+1$ if $d\alpha$ winds in the positive direction around the diagonal, and $-1$ in the opposite case. This in turn depends on whether $\gamma_i$ and $\gamma'_j$ intersect positively or negatively. In summary, we find

$$\langle \gamma_i, \gamma'_j \rangle = \text{Intersection number}(\gamma_i, \gamma'_j)$$

and we can easily compile a table of these intersection numbers. Here it is.

| | $\gamma_1'$ | $\gamma_2'$ | $\gamma_3'$ | $\gamma_4'$ | $\gamma_5'$ | $\gamma_6'$ |
|---|---|---|---|---|---|---|
| $\gamma_1$ | 0 | 0 | 0 | −1 | 0 | 0 |
| $\gamma_2$ | 0 | 0 | 0 | −1 | −1 | 0 |
| $\gamma_3$ | 0 | 0 | 0 | −1 | −1 | −1 |
| $\gamma_4$ | 1 | 1 | 1 | 0 | 0 | 0 |
| $\gamma_5$ | 0 | 1 | 1 | 0 | 0 | 0 |
| $\gamma_6$ | 0 | 0 | 1 | 0 | 0 | 0 |

EXAMPLE 3.8.12. — Let us continue the previous example, but suppose from now on that $f$ is an odd polynomial, so $f(-x) = -f(x)$, of degree $d = 2e+1$. In that case, the object $H^1_{\mathrm{perv}}(\mathbb{A}^1, f)$ is self dual via the isomorphism $\varphi\colon H^1_{\mathrm{perv}}(\mathbb{A}^1, f) \to H^1_{\mathrm{perv}}(\mathbb{A}^1, -f)$ induced by the multiplication-by-$(-1)$ map $\mathbb{A}^1 \to \mathbb{A}^1$. The Poincaré–Verdier duality pairing becomes via this isomorphism a pairing

$$H^1_{\mathrm{perv}}(\mathbb{A}^1, f) \otimes H^1_{\mathrm{perv}}(\mathbb{A}^1, f) \to \mathbb{Q}(-1) \tag{3.8.12.1}$$

which will put some constraints on the tannakian fundamental group $G$ of $H^1_{\mathrm{perv}}(\mathbb{A}^1, f)$. Let us make this constraint explicit. The basis for rapid decay homology we have considered above consists of the usual and somewhat arbitrary choice $\gamma_1, \ldots, \gamma_{2e}$ for $H_1^{\mathrm{rd}}(\mathbb{A}^1, f)$. However, we have chosen the basis $\gamma_i' = -\gamma_i$ of $H_1^{\mathrm{rd}}(\mathbb{A}^1, -f)$ in such a way that the isomorphism

$$H_1^{\mathrm{rd}}(\mathbb{A}^1, -f) \to H_1^{\mathrm{rd}}(\mathbb{A}^1, f)$$

dual to $\varphi$ sends the $\gamma_i'$ to $\gamma_i$. The pairing (3.8.11.1) can be seen via this isomorphism as an alternating bilinear form on $H_1^{\mathrm{rd}}(\mathbb{A}^1, f)$, which is in the basis $\gamma_1, \ldots, \gamma_{2e}$ given by the skew symmetric matrix

$$A = \begin{pmatrix} 0 & -{}^tT \\ T & 0 \end{pmatrix}$$

where $T \in \mathrm{GL}_e$ is the upper triangular matrix with 1's on and above the diagonal. Its coefficients are just the entries of the table of intersection numbers above. The same matrix $A$ also characterises the bilinear form in rapid decay cohomology

$$H^1_{\mathrm{rd}}(\mathbb{A}^1, f) \otimes H^1_{\mathrm{rd}}(\mathbb{A}^1, -f) \to \mathbb{Q}(-1)$$

with respect to the dual bases. This pairing is the fibre at infinity of (3.8.12.1), hence the tannakian fundamental group $G \subseteq \mathrm{GL}_{d-1}$ of $H^1_{\mathrm{perv}}(\mathbb{A}^1, f)$ must consist of matrices $g$ satisfying

$${}^tg \cdot A \cdot g = A$$

or in other words, $G$ must be contained in the symplectic group $\mathrm{Sp}_A \subseteq \mathrm{GL}_{2e}$.

CHAPTER 4

# Exponential motives

This chapter contains the technical core of our work, namely the construction of the $\mathbb{Q}$-linear neutral tannakian category $\mathbf{M}^{\exp}(k)$ of exponential motives over a subfield $k$ of $\mathbb{C}$. To this end, we first recall the basics of Nori's formalism, which attaches to a quiver representation $\rho\colon Q \to \mathbf{Vec}_{\mathbb{Q}}$ a $\mathbb{Q}$-linear abelian category $\langle Q, \rho\rangle$. We then apply this construction to a quiver consisting of tuples $[X, Y, f, n, i]$ and to the representation given by rapid decay cohomology.

## 4.1. Reminder and complements to Nori's formalism

In this section, we recall the notions of quivers and quiver representations. For us, this will just be a handy terminology to speak about categories without composition law. Classical results in the theory of quiver representations, such as the theorems of Gabriel and Kac, will play no role whatsoever.

Definition 4.1.1. — A *quiver* is the data $Q = (\mathrm{Ob}(Q), \mathrm{Mor}(Q), s, t, i)$ of two classes $\mathrm{Ob}(Q)$ and $\mathrm{Mor}(Q)$, together with maps

$$\mathrm{Mor}(Q) \;\underset{t}{\overset{s}{\rightrightarrows}}\; \mathrm{Ob}(Q), \qquad \mathrm{Ob}(Q) \xrightarrow{\;i\;} \mathrm{Mor}(Q)$$

such that $s \circ i$ and $s \circ t$ are the identity on $\mathrm{Ob}(Q)$ and, for each pair of elements $p, q \in \mathrm{Ob}(C)$, the class $\{f \in \mathrm{Mor}(Q) \mid s(f) = p, t(f) = q\}$ is a set. We say that a quiver $Q$ is *finite* if $\mathrm{Ob}(C)$ and $\mathrm{Mor}(Q)$ are both finite sets.

As the notation suggests, one regards $\mathrm{Ob}(Q)$ as a class of *objects* and $\mathrm{Mor}(Q)$ as *morphisms* between them. The maps $s$ and $t$ are the *source* and the *target* of a morphism, and each object is equipped with an *identity* morphism. From this point of view, a quiver is nothing else but a category, except that there is no composition law. We will freely adopt the terminology from category theory; for example, a *functor* from $Q$ to a category $\mathbf{C}$ is the assignment of an object of $\mathbf{C}$ to each object of $Q$ and, to each morphism in $Q$, a morphism between the corresponding objects in $\mathbf{C}$ in such a way that the identities are mapped to the identities.

Definition 4.1.2. — A *representation* of a quiver $Q$ in a category $\mathbf{C}$ is a morphism of quivers $\rho$ from $Q$ to $\mathbf{C}$. A morphism of quiver representations $(Q \xrightarrow{\rho} \mathbf{C}) \to (Q' \xrightarrow{\rho'} \mathbf{C})$ consists of a quiver morphism $\varphi\colon Q \to Q'$ and an isomorphism (a natural transformation) of quiver morphisms $\rho' \circ \psi \cong \rho$.

4.1.3. — Let $Q$ be a finite quiver, and let $\rho\colon Q \to \mathbf{C}$ be a quiver representation of $Q$ in a monoidal closed abelian category $\mathbf{C}$. The endomorphism ring $\mathrm{End}(\rho)$ is the algebra object in $\mathbf{C}$ given by

$$\mathrm{End}(\rho) = \mathrm{equaliser}\Bigg( \prod_{q\in Q} \mathrm{End}(\rho(q)) \rightrightarrows \prod_{p\to q} \mathrm{Hom}(\rho(p), \rho(q)) \Bigg) \tag{4.1.3.1}$$

where $\mathrm{End}(\rho(q))$ and $\mathrm{Hom}(\rho(p), \rho(q))$ are the internal homomorphism objects in $\mathbf{C}$. Typically, the category $\mathbf{C}$ at the receiving end of a quiver representation is the category of finite-dimensional rational vector spaces $\mathbf{Vec}_{\mathbb{Q}}$. In that case, the $\mathbb{Q}$-algebra $\mathrm{End}(\rho)$ consists of tuples $(e_q)_{q\in Q}$ of $\mathbb{Q}$-linear endomorhpisms $e_q : \rho(q) \to \rho(q)$ such that the squares

$$\begin{array}{ccc} \rho(p) & \xrightarrow{\rho(f)} & \rho(q) \\ {\scriptstyle e_p}\downarrow & & \downarrow{\scriptstyle e_q} \\ \rho(p) & \xrightarrow{\rho(f)} & \rho(q) \end{array}$$

commute for all morphisms $f\colon p \to q$ in $Q$. We may recognise (4.1.3.1) as part of a certain Hochschild simplicial complex. In particular, if $Q$ has only one object, we recognise a part of the Hochschild complex of the free $R$–algebra generated by the morphisms of $Q$ acting on the bimodule $\mathrm{End}(\rho(q))$. The Hochschild cohomology vanishes from $H^2$ on, and the first Hochschild cohomology group, whose elements have the interpretation of derivations modulo inner derivations, is the coequaliser of (4.1.3.1).

Given an arbitrary quiver $Q$, a representation $\rho\colon Q \to \mathbf{C}$ in a closed monoidal category $\mathbf{C}$ and a finite subquiver $Q_0 \subseteq Q$, we can consider the algebra of endomorphisms $E_0 := \mathrm{End}_{\mathbb{Q}}(\rho|_{Q_0})$ as before. It is an algebra object in $\mathbf{C}$. The endomorphism algebra $\mathrm{End}(\rho)$ is the formal limit of algebra objects

$$\mathrm{End}(\rho) = \lim_{Q_0 \subseteq Q} \mathrm{End}(\rho|_{Q_0})$$

as $Q_0$ runs over the finite subquivers of $Q$ and transition maps are restrictions. Thus, $\mathrm{End}(\rho)$ is a pro-object in the category of algebra objects in $\mathbf{C}$. The following lemma tells us that in the case $\mathbf{C} = \mathbf{Vec}_{\mathbb{Q}}$ case we don't have to worry about the distinction between formal pro-objects the category of finite-dimensional algebras and infinite-dimensional algebras equipped with a topology.

Lemma 4.1.4. — *Let $I$ be a partially ordered set (where for every two elements $i, j \in I$ there exists $k \in I$ with $k \geqslant i$ and $k \geqslant j$), and let $(E_i)_{i\in I}$ be a collection of finite-dimensional $\mathbb{Q}$-algebras, together with algebra morphisms $r_{ji} : E_j \to E_i$ for $j \geqslant i$ satisfying $r_{ji} \circ r_{kj} = r_{ki}$ for $k \geqslant j \geqslant i$. Set*

$$E := \lim_{i\in I} E_i = \Big\{ (e_i)_{i\in I} \in \prod_{i\in I} E_i \;\Big|\; r_{ij}(e_j) = e_i \textit{ for all } j > i \Big\}$$

*and denote by* $p_i : E \to E_i$ *the canonical projections. For every finite-dimensional* $\mathbb{Q}$*-algebra* $F$*, the canonical map*

$$\operatorname*{colim}_{i\in I} \operatorname{Hom}_{\mathbb{Q}-\mathrm{alg}}(E_i, F) \xrightarrow{(*)} \operatorname*{colim}_{i\in I} \operatorname{Hom}_{\mathbb{Q}-\mathrm{alg}}(E/\ker(p_i), F)$$

*is bijective.*

Proof. Any element of the left hand set is represented by an algebra morphism $h\colon E_i \to F$ for some $i \in I$, and the map labelled $(*)$ sends this element to the class of the composite of $h$ with the canonical injection $u_i : E/\ker(p_i) \to E_i$.

*Injectivity.* Any two elements of the left hand set can be represented by algebra morphisms $g : E_i \to F$ and $h\colon E_i \to F$ for some large enough $i \in I$. To say that $h$ and $g$ are mapped to the same element by $(*)$ is to say that there exists an element $j \geqslant i$ such that the two composite maps

$$E/\ker(p_j) \to E/\ker(p_i) \xrightarrow{u_i} E_i \xrightarrow{g,\,h} F$$

coincide. Here $E/\ker(p_j) \to E/\ker(p_i)$ is the canonical projection obtained from $\ker(p_j) \subseteq \ker(p_i)$. These maps are the same as the composite maps

$$E/\ker(p_j) \xrightarrow{u_i} E_j \xrightarrow{r_{ji}} E_i \xrightarrow{g,\,h} F$$

which means that $g \circ r_{ji}$ coincides with $g \circ r_{ji}$ on the image of the projection $E \to E_j$. Since $E_j$ is finite-dimensional as a $\mathbb{Q}$-vector space, the image of the projection $E \to E_j$ is equal to the image of $r_{kj} : E_k \to E_j$ for some $k \geqslant j$. Hence the maps $g \circ r_{ki}$ and $h \circ r_{ki}$ from $E_k$ to $F$ are equal, which means that $g$ and $h$ represent the same element.

*Surjectivity.* Pick an algebra homomorphism $h\colon E/\ker(p_i) \to F$ representing an element of the right hand set. Since $E_i$ is finite-dimensional as a $\mathbb{Q}$-vector space, there exists $j \geqslant i$ such that the image of $p_i : E \to E_i$ is equal to the image of $r_{ji} : E_j \to E_i$. The composite

$$E_j \to E_j/\ker(r_{ji}) \cong E/\ker(p_i) \xrightarrow{h} F$$

represents a preimage by $(*)$ of the class of $h$. □

4.1.5. — Let us keep the notation from Lemma 4.1.4. The collection of the algebras $E_i$ and morphisms $r_{ji}$ describes a pro-object in the category of finite-dimensional algebras. Elements of the set $\operatorname{colim} \operatorname{Hom}_{\mathbb{Q}-\mathrm{alg}}(E_i, F)$ are morphisms of pro-objects from $(E_i, r_{ji})$ to $F$. On the other hand, we can define a topology on $E$ by declaring the ideals $\ker(p_i)$ to be a fundamental system of open neighbourhoods of 0. Elements of the set $\operatorname{colim} \operatorname{Hom}_{\mathbb{Q}-\mathrm{alg}}(E/\ker(p_i), F)$ are then the same as continuous algebra morphisms $E \to F$ for the discrete topology on $F$. A consequence of the lemma is that the category of finite-dimensional $(E_i, r_{ji})$-modules is the same as the category of finite-dimensional continuous $E$-modules. The statement of the lemma, as well as the latter consequence of it, are false if instead of finite-dimensional algebras over a field one takes finite $R$-algebras over a coherent ring $R$, even for $R = \mathbb{Z}$.

Definition 4.1.6. — Let $Q$ be a quiver and $\rho\colon Q \to \mathbf{Vec}_{\mathbb{Q}}$ a quiver representation. We call *linear hull* of $(Q, \rho)$ the category $\langle Q, \rho\rangle$ defined as follows:

(1) Objects of $\langle Q, \rho\rangle$ are triples $(M, Q_0, \alpha)$ consisting of a finite-dimensional $\mathbb{Q}$-vector space $M$, a finite subquiver $Q_0$ of $Q$, and a $\mathbb{Q}$-linear action $\alpha$ of the algebra $\mathrm{End}(\rho|_{Q_0})$ on $M$.

(2) Morphisms $(M_1, Q_1, \alpha_1) \to (M_2, Q_2, \alpha_2)$ in $\langle Q, \rho\rangle$ are linear maps $f\colon M_1 \to M_2$ with the property that there exists a finite subquiver $Q_3$ of $Q$ containing $Q_1$ and $Q_2$ such that $f$ is $\mathrm{End}(\rho|_{Q_3})$-linear. The action of $\mathrm{End}(\rho|_{Q_3})$ on $M_i$ is obtained via $\alpha_i$ and the restriction $\mathrm{End}(\rho|_{Q_3}) \to \mathrm{End}(\rho|_{Q_i})$.

(3) Composition of morphisms in $\langle Q, \rho\rangle$ is composition of linear maps.

Equivalently, in light of Lemma 4.1.4, the linear hull $\langle Q, \rho\rangle$ is the category of continuous $\mathrm{End}(\rho)$-modules which are finite-dimensional as vector spaces. It is therefore a $\mathbb{Q}$-linear abelian category. We call *canonical lift* the representation

$$\widetilde{\rho}\colon Q \to \langle Q, \rho\rangle$$

sending an object $q \in Q$ to the triple $\widetilde{\rho}(q) = (\rho(q), \{q\}, \mathrm{id})$ and a morphism $p \to q$ to the linear map $\rho(f)\colon \rho(p) \to \rho(q)$.

Proposition 4.1.7. — *Let $\rho\colon Q \to \mathbf{Vec}_{\mathbb{Q}}$ be a quiver representation. Every object of the abelian category $\langle Q, \rho\rangle$ is isomorphic to a subquotient of a finite sum of objects of the form $\tilde{\rho}(q)$ for $q$ in $Q$.*

Proof. Let $M$ be an object of $\langle Q, \rho\rangle$, that is, a finite-dimensional vector space together with an $E_0$-module structure for some finite subquiver $Q_0 \subseteq Q$ and $E_0 := \mathrm{End}(\rho|_{Q_0})$. We can regard $E_0$ with its left $E_0$-module structure as an object of $\langle Q, \rho\rangle$ too. Since $M$ is finite-dimensional, there is a surjection of $E_0$-modules $E_0^n \to M$ for some integer $n \geqslant 0$, hence it suffices to prove the proposition in the case $M = E_0$. There is an exact sequence of $E_0$-modules

$$0 \longrightarrow E_0 \longrightarrow \prod_{q \in Q_0} \mathrm{End}(\rho(q)) \longrightarrow \prod_{p \to q} \mathrm{Hom}(\rho(p), \rho(q)) \tag{4.1.7.1}$$

which shows that $E_0$, seen as a left $E_0$-module, is indeed isomorphic to a subobject of a product of modules of the form $\rho(q)$. Notice that $\mathrm{End}(\rho(q))$ is isomorphic as an $E_0$-module to $\rho(q)^d$ for $d = \dim_{\mathbb{Q}}(\rho(q))$. □

4.1.8. — An important feature of linear hulls of quiver representations is that they are functorial in the following sense: Given a morphism of quiver representations, that is, a triangle of quiver morphisms together with a natural transform

$$\begin{array}{ccc} Q & \xrightarrow{\varphi} & Q' \\ & {\scriptstyle\rho}\searrow \quad \swarrow{\scriptstyle\rho'} & \\ & \mathbf{Vec}_{\mathbb{Q}} & \end{array} \qquad \begin{array}{ll} s: & \rho' \circ \varphi \xrightarrow{\cong} \rho \\ s_q: & \rho'(\varphi(q)) \xrightarrow{\cong} \rho(q) \end{array}$$

we obtain a functor $\Phi : \langle Q, \rho\rangle \to \langle Q', \rho'\rangle$ by setting $\Phi(M, Q_0, \alpha) = (M, \varphi(Q_0), \alpha \circ \sigma)$, where $\varphi(Q_0)$ is the image of the finite subquiver $Q_0 \subseteq Q$ in $Q'$ under $\varphi$, and $\sigma$ the morphism of algebras $\mathrm{End}(\rho'|_{\varphi(Q_0)}) \to \mathrm{End}(\rho|_{Q_0})$ obtained from $s$. In terms of 4.1.3, the morphism $\sigma$ sends the tuple $(e_{q'})_{q' \in \varphi(Q_0)}$ to the tuple $(s_q \circ e_{\varphi(q)} \circ s_q^{-1})_{q \in Q_0}$. We notice that the functor $\Phi$ is faithful and exact, and that it commutes with the forgetful functors and up to natural isomorphisms with the canonical lifts.

4.1.9. — The induced functor $\Phi$ in the previous paragraph depends naturally on the morphism of quiver representations $(\varphi, s)$ in the following sense. Let $\rho\colon Q \to \mathbf{Vec}_{\mathbb{Q}}$ and $\rho' : Q' \to \mathbf{Vec}_{\mathbb{Q}}$ be quiver representations, and let

$$\begin{array}{ccc} Q & \overset{\varphi}{\underset{\psi}{\rightrightarrows}} & Q' \\ & {\scriptstyle\rho}\searrow \quad \swarrow{\scriptstyle\rho'} & \\ & \mathbf{Vec}_{\mathbb{Q}} & \end{array} \qquad\qquad \begin{array}{l} s : \rho' \circ \varphi \to \rho \\[1em] t : \rho' \circ \psi \to \rho \end{array}$$

be two morphisms of quiver representations. Denote by $\Phi$ and $\Psi$ the induced functors between linear hulls $\langle Q, \rho\rangle \to \langle Q', \rho'\rangle$. We call *2–morphism* from $(\varphi, s)$ to $(\psi, t)$ every natural transform $\eta : \widetilde{\rho}' \circ \varphi \to \widetilde{\rho}' \circ \psi$ with the property that for every $q \in Q$ the diagram of $R$–modules

$$\begin{array}{ccc} \rho'(\varphi(q)) & \xrightarrow{\eta_q} & \rho'(\psi(q)) \\ \Big\downarrow{\scriptstyle s_q} & & \Big\downarrow{\scriptstyle t_q} \\ \rho(q) & = & \rho(q) \end{array} \tag{4.1.9.1}$$

commutes. Such a 2–morphism $\eta$ indeed induces a morphism of functors $E : \Phi \to \Psi$, namely, for every object $X = (M, F, \alpha)$ in $\langle Q, \rho\rangle$, the morphism

$$E_X : \Phi(X) = (M, \varphi(F), \alpha \circ \sigma) \to \Psi(X) = (M, \psi(F), \alpha \circ \tau)$$

in $\langle Q', \rho'\rangle$ given by the identity $\mathrm{id}_M$. Let us check that $\mathrm{id}_M : \Phi(X) \to \Psi(X)$ is indeed a morphism in $\langle Q', \rho'\rangle$. We can without loss of generality suppose that $Q$ and $Q'$ are finite quivers. What has to be shown is that the two actions of $\mathrm{End}(\rho')$ on $M$, one induced by $s$ and the other by $t$, agree. Indeed, already the two algebra morphisms

$$\sigma, \tau : \mathrm{End}(\rho') \to \mathrm{End}(\rho)$$

are the same: given an element $(e_{q'})_{q' \in Q'}$ of $\mathrm{End}(\rho')$ and $q \in Q$, the diagram

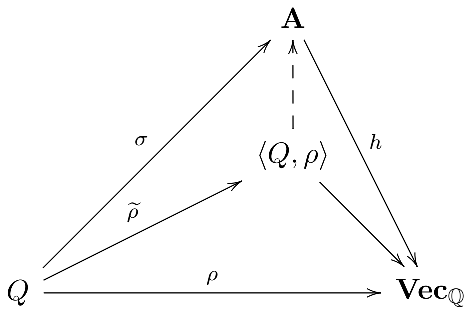
$$\begin{array}{ccccccc} \rho(q) & \xrightarrow{s_q^{-1}} & \rho'(\varphi(q)) & \xrightarrow{e_{\varphi(q)}} & \rho'(\varphi(q)) & \xrightarrow{s_q} & \rho(q) \\ \Big\| & & {\scriptstyle\eta_q}\Big\downarrow & & {\scriptstyle\eta_q}\Big\downarrow & & \Big\| \\ \rho(q) & \xrightarrow{t_q^{-1}} & \rho'(\psi(q)) & \xrightarrow{e_{\psi(q)}} & \rho'(\psi(q)) & \xrightarrow{t_q} & \rho(q) \end{array}$$

commutes because $\eta_q$ is not just an arbitrary morphism of modules, but comes from a morphism $\widetilde{\rho}'(\varphi(q)) \to \widetilde{\rho}'(\psi(q))$ in $\langle Q', \rho'\rangle$ and hence is $\mathrm{End}(\rho')$–linear.

THEOREM 4.1.10. — *Let $\mathbf{A}$ be an abelian, $\mathbb{Q}$–linear category, and let $h\colon \mathbf{A} \to \mathbf{Vec}_{\mathbb{Q}}$ be a faithful, linear and exact functor. Regard $h$ as a quiver representation. The canonical lift $\widetilde{h}\colon \mathbf{A} \to \langle \mathbf{A}, h\rangle$ is an equivalence of categories.*

References. In a slightly different form, the statement goes back to Freyd and Mitchell, who proved their embedding theorem for abelian categories in 1964. In the form presented here, Theorem 4.1.10 was originally shown by Nori in [**Nor**]. There are accounts by Bruguières, Levine, and Huber and Müller-Stach ([**Bru04, Lev05, HM14**]). Ivorra deduces in [**Ivo17**] the result from a more general construction. □

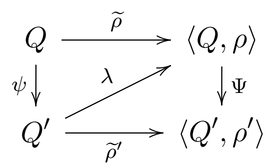
Theorem 4.1.11 (Nori's universal property). — *Let* $\mathbf{A}$ *be a* $\mathbb{Q}$*-linear abelian category, together with a functor* $\sigma\colon Q \to \mathbf{A}$*, and let* $h\colon \mathbf{A} \to \mathbf{Vec}_{\mathbb{Q}}$ *be an exact,* $\mathbb{Q}$*-linear and faithful functor such that the following diagram of solid arrows commutes.*




*Then the above dashed arrow, rendering the whole diagram commutative, exists and is unique up to a unique isomorphism.*

Proof. We can then regard $\sigma$ as a morphism of quiver representations from $\rho\colon Q \to \mathbf{Vec}_{\mathbb{Q}}$ to $h\colon \mathbf{A} \to \mathbf{Vec}_{\mathbb{Q}}$. By naturality of the linear hull construction it gives a functor $\langle Q, \rho\rangle \to \langle \mathbf{A}, h\rangle$, or, in view of theorem 4.1.10, a functor $\langle Q, \rho\rangle \to \mathbf{A}$ which renders the whole diagram commutative up to natural isomorphisms. □

Lemma 4.1.12. — *Let* $\psi : (Q \xrightarrow{\rho} \mathbf{Vec}_{\mathbb{Q}}) \to (Q' \xrightarrow{\rho'} \mathbf{Vec}_{\mathbb{Q}})$ *be a morphism of quiver representations. The induced functor* $\Psi : \langle Q, \rho\rangle \to \langle Q', \rho'\rangle$ *is an equivalence of categories if and only if there exists a quiver representation* $\lambda : Q' \to \langle Q, \rho\rangle$ *such that the following diagram commutes up to natural isomorphisms.*




(4.1.12.1)

Proof. If $\Psi$ is an equivalence of categories, then there exists a functor $\Phi : \langle Q', \rho'\rangle \to \langle Q, \rho\rangle$ and isomorphisms $\Phi \circ \Psi \cong \mathrm{id}$ and $\Psi \circ \Phi \cong \mathrm{id}$. A possible choice for $\lambda$ is then $\lambda := \Phi \circ \widetilde{\rho}'$, indeed, since the outer square in (4.1.12.1) commutes up to an isomorphism, we have isomorphisms

$$\Psi \circ \lambda \cong \Psi \circ \Phi \circ \widetilde{\rho}' \cong \widetilde{\rho}' \qquad \text{and} \qquad \lambda \circ \psi = \Phi \circ \widetilde{\rho}' \circ \psi \cong \Phi \circ \Psi \circ \widetilde{\rho} \cong \widetilde{\rho}$$

as required.

On the other hand, suppose that a representation $\lambda$ as in the statement of the lemma exists. We extend the diagram (4.1.12.1) to a diagram

$$\begin{array}{ccc} Q & \xrightarrow{\widetilde{\rho}} & \langle Q, \rho\rangle \\ {\scriptstyle\psi}\downarrow & \nearrow^{\lambda} & \downarrow{\scriptstyle\Psi} \\ Q' & \xrightarrow{\widetilde{\rho}'} & \langle Q', \rho'\rangle \\ {\scriptstyle\lambda}\downarrow & \nearrow^{\Psi} & \downarrow{\scriptstyle\Lambda} \\ \langle Q, \rho\rangle & \xrightarrow[P]{} & \langle\langle Q, \rho\rangle, f\rangle \end{array} \tag{4.1.12.2}$$

with arrows as follows: Let $f\colon \langle Q, \rho\rangle \to \mathbf{Vec}_{\mathbb{Q}}$ and $f' : \langle Q', \rho'\rangle \to \mathbf{Vec}_{\mathbb{Q}}$ be the forgetful functors. We have an isomorphism $f' \circ \Psi \cong f$, hence an isomorphism $\rho' = f' \circ \widetilde{\rho}' \cong f' \circ \Psi \circ \lambda \cong f \circ \lambda$, and can thus view $\lambda$ as a morphism of quiver representations from $\rho'$ to $f$. The arrow $\Lambda$ is the corresponding functor. The functor $P$ is the canonical lift of $\widetilde{\rho}$ regarded as a morphism of quiver representations from $\rho$ to $f$; by Theorem 4.1.10 it is an equivalence of categories. Let $\iota$ be a quasi–inverse to $P$. I claim that the functor $\Phi := \iota \circ \Lambda$ is a quasi–inverse to $\Psi$.

To get an isomorphism $\Phi \circ \Psi \cong \mathrm{id}$ it suffices to get an isomorphism $\Lambda \circ \Psi \cong P$. Let us apply 4.1.9 to the representations

$$\begin{array}{ccc} Q & \overset{\lambda\circ\psi}{\underset{\widetilde{\rho}}{\rightrightarrows}} & \langle Q, \rho\rangle \\ & {\scriptstyle\rho}\searrow \quad \swarrow{\scriptstyle f} & \\ & \mathbf{Vec}_{\mathbb{Q}} & \end{array} \qquad \begin{array}{l} f \circ \lambda \circ \psi \cong f \circ \widetilde{\rho} = \rho \\[1em] f \circ \widetilde{\rho} = \rho \end{array}$$

where we use an isomorphism $\lambda\circ\psi \cong \widetilde{\rho}$ which makes (4.1.12.1) commute. This isomorphism induces an isomorphism $\eta : \widetilde{f} \circ \lambda \circ \psi \cong \widetilde{f} \circ \widetilde{\rho}$ which makes the diagrams corresponding to (4.1.9.1) commute, hence we obtain an isomorphism of functors $\Lambda \circ \Psi \cong P$. It remains to construct an isomorphism $\Psi \circ \Phi \cong \mathrm{id}$. This is done by replacing the diagram (4.1.12.1) in the statement of the Lemma with the bottom half of (4.1.12.2), and the same application of 4.1.9. □

4.1.13 (**Caveat**). — In the situation of Lemma 4.1.12, it will not do to just produce a representation $\lambda$ as in diagram (4.1.12.1) and natural isomorphisms of $R$–modules $\rho(q) \cong f(\lambda(q))$ in order to show that $\Psi$ is an equivalence. Such a $\lambda$ will produce *some* functor $\Phi : \langle Q', \rho'\rangle \to \langle Q, \rho\rangle$ which, in general, is not a quasi-inverse to $\Psi$. Each time we apply 4.1.12, the hard part is not to define $\lambda$, but to check commutativity of the diagram. The point seems to have been overlooked at several places[1]. Consider for example a homomorphism of finite groups $G' \to G$, the quivers $Q$ and $Q'$ of finite $G$-sets, respectively $G'$-sets, and the quiver representations $\rho$ and $\rho'$ which associate with a set $X$ the vector space generated by $X$. The linear hulls identify with the categories of $\mathbb{Q}$-linear group representations, and the restriction functor $Q \to Q'$ is a morphism of quiver representations which induces the restriction functor between representation categories. For any $G'$-set $X'$ write

[1]In [**HM14**], the proof of Corollary 1.7 is incomplete because of this problem, as is Arapura's [**Ara13**] proof of Theorem 4.4.2. Levine [**Lev05**] cites Nori's [**Nor**], where Nori draws the right diagram but does not show that it commutes.

$\lambda(X)$ for the trivial $G$-representation on the vector space generated by the set $X$. We obtain a diagram

$$\begin{array}{ccc} G\,\mathbf{Set} & \xrightarrow{\text{free}} & \mathbf{Rep}_{\mathbb{Q}}(G) \\ {\scriptstyle \psi=\text{res}}\downarrow & \nearrow^{\lambda} & \downarrow{\scriptstyle \Psi=\text{res}} \\ G'\,\mathbf{Set} & \xrightarrow[\text{free}]{} & \mathbf{Rep}_{\mathbb{Q}}(G') \end{array}$$

which does in not commute except in trivial cases, but commutes after forgetting the group actions. The functor $\Psi$ is not an equivalence, trivial cases excepted, and the functor induced by $\lambda$ sends a $G'$–representation $V$ to the constant $G$–representation with underlying module $V$.

Definition 4.1.14. — Let $\rho\colon Q \to \mathbf{Vec}_{\mathbb{Q}}$ and $\rho' : Q' \to \mathbf{Vec}_{\mathbb{Q}}$ be quiver representations. We denote by

$$\rho \boxtimes \rho' : Q \boxtimes Q' \to \mathbf{Vec}_{\mathbb{Q}}$$

the following quiver representation. Objects of the quiver $Q \boxtimes Q'$ are pairs $(q, q')$ consisting of an object $q$ of $Q$ and an object $q'$ of $Q'$, and morphisms are either of the form $(\mathrm{id}_q, f') : (q, q') \to (p, p')$ for some morphism $f' : q' \to p'$ in $Q'$, or of the form $(f, \mathrm{id}_{q'}) : (q, q') \to (p, q')$ for some morphism $f\colon p \to q$ in $Q$. The representation $\rho \boxtimes \rho'$ is defined by

$$(\rho \boxtimes \rho')(q, q') = \rho(q) \otimes_R \rho'(q')$$

on objects, and by $(\rho\boxtimes\rho')(\mathrm{id}_q, f') = \mathrm{id}_{\rho(q)} \otimes \rho(f')$ and $(\rho\boxtimes\rho')(f, \mathrm{id}_{q'}) = \rho(f)\otimes\mathrm{id}_{\rho'(q')}$ on morphisms.

4.1.15. — Our next proposition relates the linear hull of a quiver representation $\rho \boxtimes \rho'$ with the tensor product of the linear hulls of $\rho$ and $\rho'$. A tensor product $\mathbf{A}\otimes\mathbf{B}$ of abelian $\mathbb{Q}$–linear categories $\mathbf{A}$ and $\mathbf{B}$, as introduced in [**Del90**], is a $\mathbb{Q}$-linear category characterised up to equivalence by a universal property. It does not exist in general as is shown in [**Lop13**], however, it exists and has good properties as soon as one works in an appropriate enriched setting, as is shown in [**Gre10**]. We only need to know the following fact: If $\mathbf{A}$ is the category of continuous, finite-dimensional $A$-modules and $\mathbf{B}$ the category of continuous, finite-dimensional $B$-modules for some $\mathbb{Q}$-profinite algebras

$$A = \lim_i A_i \qquad \text{and} \qquad B = \lim_j B_j$$

then $\mathbf{A} \otimes \mathbf{B}$ exists and is given by the category of continuous $A\widehat{\otimes}B$–modules, where

$$A\widehat{\otimes}B := \lim_{i,j} A_i \otimes B_j$$

stands for the completed tensor product. This follows from §5.1 and Proposition 5.3 of [**Del90**].

Proposition 4.1.16. — *There is a canonical faithful and exact functor*

$$\langle \rho \boxtimes \rho', Q \boxtimes Q' \rangle \to \langle Q, \rho \rangle \otimes \langle Q', \rho' \rangle$$

*which commutes with the forgetful functors to $\mathbf{Vec}_{\mathbb{Q}}$, and is natural in $\rho$ and $\rho'$ for morphisms of quiver representations. This functor is an equivalence of categories.*

Proof. It suffices to construct a functor in the case where $Q$ and $Q'$ are finite quivers. Set

$$V := \bigoplus_{q\in Q} \rho(q)$$

and write $X \subseteq \mathrm{End}_R(V)$ for the finite set of compositions of the form $V \xrightarrow{\mathrm{pr}} \rho(p) \xrightarrow{\rho(f)} \rho(q) \xrightarrow{\subseteq} V$ for some morphism $f$ in $Q$, and write $E_X := \mathrm{End}(\rho) \subseteq \mathrm{End}(V)$ for the commutator of $X$. Define $E_{X'} \subseteq \mathrm{End}(V')$ and $E_{X\boxtimes X'} \subseteq \mathrm{End}(V \otimes V')$ similarly. We want to show that the canonical, natural morphism of $\mathbb{Q}$–algebras $E_X \otimes_{\mathbb{Q}} E_{X'} \to E_{X\boxtimes X'}$ given in the diagram

$$\begin{array}{ccc} E_X \otimes_{\mathbb{Q}} E_{X'} & \longrightarrow & E_{X\boxtimes X'} \\ \big\downarrow & & \big\downarrow{\scriptstyle\subseteq} \\ \mathrm{End}(V) \otimes_{\mathbb{Q}} \mathrm{End}(V') & \xrightarrow{\alpha} & \mathrm{End}(V \otimes V') \end{array} \qquad \alpha(f \otimes f')(v \otimes v') = f(v) \otimes f'(v') \quad (4.1.16.1)$$

is an isomorphism. All morphisms in this diagram are injective, and $\alpha$ is an isomorphism. We want to show that the top horizontal map is surjective. Let $f \in \mathrm{End}(V \otimes V')$ be an endomorphism that commutes with $X \boxtimes X'$. We write $f$ as $f = \alpha(f_1 \otimes f_1' + \cdots + f_n \otimes f_n')$ with $f_i \in \mathrm{End}(V)$ and linearly independent $f_i' \in \mathrm{End}(V')$. For all $x \in X$ we have $(x \otimes 1) \circ f = f \circ (x \otimes 1)$, that is,

$$\sum_{i=1}^{n} (x \circ f_i - f_i \circ x) \otimes f_i' = 0$$

and hence $f_i \in E_X$. In other words, $f$ comes via $\alpha$ from an element of $E_X \otimes \mathrm{End}(V')$, and symetrically, $f$ comes from an element of $\mathrm{End}(V) \otimes E_{X'}$. Finally, again since $E_X$ and $E_{X'}$ are direct factors of $\mathrm{End}(V)$ and $\mathrm{End}(V')$, we have

$$\big(E_X \otimes \mathrm{End}(V')\big) \cap \big(\mathrm{End}(V) \otimes E_{X'}\big) = E_X \otimes_R E_{X'}$$

so $\alpha^{-1}(f)$ is indeed an element of $E_X \otimes_R E_{X'}$ as we wanted to show. □

4.1.17. — All statements presented in this section hold verbatim for $R$-linear quiver representations when $R$ is a field. With the exception of Lemma 4.1.4 and Proposition 4.1.16 one can even take for $R$ a commutative coherent ring, and replace categories of finite-dimensional vector spaces by categories of finitely presented modules. If in Proposition 4.1.16 we choose to work with a coherent ring of coefficients $R$, the exact and faithful functor still exists, but it is in general not an equivalence of categories. A sufficient condition for this functor to be an equivalence of categories is that $R$ is a hereditary ring, and $\rho(q)$ and $\rho'(q')$ are projective $R$–modules for all $q \in Q$ and $q' \in Q'$. *Hereditary* means: Every ideal of $R$ is projective, or equivalently, every submodule of a projective module is projective. Fields, finite products of fields, Dedekind rings and finite rings are examples. A commutative, coherent and hereditary ring which has no zero divisors is either a field or a Dedekind ring.

One might be tempted to replace the category $\mathbf{Vec}_{\mathbb{Q}}$ in Definition 4.1.6 by an arbitrary abelian monoidal closed category. However, this will not result in a useful definition, since Theorem 4.1.10 and the universal property described in 4.1.11 do not hold in this generality. The point is the following: Let $\rho\colon Q \to \mathbf{Vec}_{\mathbb{Q}}$ be a quiver representation, and regard the forgetful functor $f\colon \langle Q, \rho\rangle \to \mathbf{Vec}_{\mathbb{Q}}$ as a quiver representation. The the key turn in the proof of 4.1.10 is to show that the canonical lift $\widetilde{f}$ of $f$, and the functor $P$ induced by $\widetilde{\rho}\colon Q \to \langle Q, \rho\rangle$ viewed as a morphism of quiver representations

$$\widetilde{f},\, P \;:\; \langle Q,\rho\rangle \rightrightarrows \langle\langle Q,\rho\rangle, f\rangle$$

are isomorphic functors. This relies on the fact that the neutral object for the tensor product in $\mathbf{Vec}_{\mathbb{Q}}$ is a projective generator, which is particular to categories of modules. In the case where we replace $\mathbf{Vec}_{\mathbb{Q}}$ by a tannakian category, a correct abelian hull which satisfies Ivorra's universal property (it is the initial object in a certain strict 2–category, see [**Ivo17**], Definition 2.2) is given by the equaliser category of $\widetilde{f}$ and $P$.

## 4.2. Exponential motives

We fix for this section a field $k$ endowed with a complex embedding $\sigma\colon k \hookrightarrow \mathbb{C}$. All varieties and morphisms of varieties are understood to be defined over $k$. Given a variety $X$, a closed subvariety $Y$ of $X$, and a regular function $f$ on $X$, when there is no risk of confusion, we will still denote by $X, Y, f$ the associated complex analytic varieties $X(\mathbb{C}), Y(\mathbb{C})$, and the holomorphic function $f_{\mathbb{C}}\colon X(\mathbb{C}) \to \mathbb{C}$.

Definition 4.2.1. — Let $k$ be a field. The *quiver of exponential relative varieties over* $k$ is the quiver $\mathrm{Q}^{\exp}(k)$ consisting of the following objects and morphisms:

(1) Objects are tuples $[X, Y, f, n, i]$, where $X$ is a variety over $k$, $Y \subseteq X$ is a closed subvariety, $f$ is a regular function on $X$, and $n$ and $i$ are integers.

(2) Morphisms with target $[X, Y, f, n, i]$ are given by either (a), (b) or (c) as follows:
   - (a) a morphism $h^*\colon [X', Y', f', n, i] \to [X, Y, f, n, i]$ for each morphism of varieties $h\colon X \to X'$ satisfying $h(Y) \subseteq Y'$ and $f' \circ h = f$,
   - (b) a morphism $\partial\colon [Y, Z, f|_Y, n-1, i] \to [X, Y, f, n, i]$ for each pair of closed immersions $Z \subseteq Y \subseteq X$,
   - (c) a morphism $[X \times \mathbb{G}_m, (Y \times \mathbb{G}_m) \cup (X \times \{1\}), f \boxplus 0, n+1, i+1] \to [X, Y, f, n, i]$.

We refer to the integer $n$ as *cohomological degree* of just *degree*, and to the integer $i$ as *twist*.

Definition 4.2.2. — The *Betti representation* of the quiver of exponential relative varieties over $k$ is the functor $\rho\colon \mathrm{Q}^{\exp}(k) \to \mathbf{Vec}_{\mathbb{Q}}$ defined on objects by

$$\rho([X, Y, f, n, i]) = H^n_{\mathrm{rd}}(X, Y, f)(i),$$

where $(i)$ denotes the tensor product with the $(-i)$–fold tensor power of the one-dimensional vector space $H^1(\mathbb{G}_m, \mathbb{Q})$, and as follows on morphisms:

(a) a morphism of type (a) is sent to the map

$$H^n_{\mathrm{rd}}(X', Y', f')(i) \to H^n_{\mathrm{rd}}(X, Y, f)(i)$$

induced, by passing to the limit $r \to +\infty$, from functoriality of relative cohomology with respect to the morphism of pairs $h\colon [X, Y \cup f^{-1}(S_r)] \to [X', Y' \cup (f')^{-1}(S_r)]$;

(b) a morphism of type (b) is sent to the map

$$H^{n-1}_{\mathrm{rd}}(Y, Z, f_{|Y})(i) \to H^n_{\mathrm{rd}}(X, Y, f)(i)$$

induced, by passing to the limit $r \to +\infty$, from the composition

$$\begin{array}{ccc} H^{n-1}(Y, Z \cup f^{-1}_{|Y}(S_r)) & & \\ \downarrow \wr & & \\ H^{n-1}(Y \cup f^{-1}(S_r), Z \cup f^{-1}(S_r)) & \longrightarrow & H^n(X, Y \cup f^{-1}(S_r)) \end{array}$$

where the horizontal map is the connecting morphism in the long exact sequence associated with the triple $Z \cup f^{-1}(S_r) \subseteq Y \cup f^{-1}(S_r) \subseteq X$, and the vertical morphism is the inverse of the map induced by the obvious morphism of pairs, which is an isomorphism by excision;

(c) a morphism of type (c) is sent to the map

$$H^{n+1}_{\mathrm{rd}}(X \times \mathbb{G}_m, (Y \times \mathbb{G}_m) \cup (X \times \{1\}), f \boxplus 0)(i+1) \to H^n_{\mathrm{rd}}(X, Y, f)(i)$$

induced by the Künneth isomorphism (Proposition 3.1.1)

$$\begin{aligned} H^{n+1}_{\mathrm{rd}}(X \times \mathbb{G}_m, &(Y \times \mathbb{G}_m) \cup (X \times \{1\}), f \boxplus 0) \\ &\xrightarrow{\sim} H^n_{\mathrm{rd}}(X, Y, f) \otimes H^1(\mathbb{G}_m, \{1\}) = H^n_{\mathrm{rd}}(X, Y, f)(-1). \end{aligned}$$

Definition 4.2.3. — The category of *exponential motives* over $k$ is the linear hull

$$\mathbf{M}^{\exp}(k) = \langle \mathrm{Q}^{\exp}(k), \rho \rangle,$$

that is, the category whose objects are triples $(M, Q, \alpha)$, where $M$ is a finite-dimensional rational vector space, $\mathrm{Q} \subseteq \mathrm{Q}^{\exp}(k)$ a finite subquiver, and $\alpha$ a linear action of $\mathrm{End}(\rho|_Q)$ on $M$. We write

$$R_B\colon \mathbf{M}^{\exp}(k) \longrightarrow \mathbf{Vec}_{\mathbb{Q}}$$

for the forgetful functor, and call it *Betti realisation*. Given an object $[X, Y, f, n, i]$ of the quiver $\mathrm{Q}^{\exp}(k)$, we denote by $H^n(X, Y, f)(i)$ the exponential motive $\widetilde{\rho}([X, Y, f, n, i])$. Whenever $Y = \varnothing$ or $i = 0$, we shall usually omit them from the notation.

4.2.4. — Let us list for future reference a few conspicuous properties of the category $\mathbf{M}^{\exp}(k)$. First of all, $\mathbf{M}^{\exp}(k)$ is an abelian and $\mathbb{Q}$-linear category, and there is by definition a commutative diagram

$$\begin{array}{ccc} & \mathbf{M}^{\exp}(k) & \\ {\scriptstyle \widetilde{\rho} = \text{motive of}} \nearrow & & \searrow {\scriptstyle R_B = \text{Betti realisation}} \\ \mathrm{Q}^{\exp}(k) & \xrightarrow{\quad \rho = \text{rapid decay coho.} \quad} & \mathbf{Vec}_{\mathbb{Q}} \end{array}$$

where $\rho$ and its canonical lift $\widetilde{\rho}$ are quiver representations, and where $R_B$ is a faithful, exact and *conservative* functor. Conservative means that a morphism $f$ in $\mathbf{M}^{\exp}(k)$ is an isomorphism if and only if its Betti realisation $R_B(f)$ is an isomorphism of vector spaces. From Proposition 4.1.7 we know that every object in $\mathbf{M}^{\exp}(k)$ is isomorphic to a subquotient of a sum of objects of the form $H^n(X, Y, f)(i)$. Morphisms in the quiver $\mathrm{Q}^{\exp}(k)$ lift to morphisms in $\mathbf{M}^{\exp}(k)$. In particular we have morphisms of motives

$$h^*\colon H^n(X', Y', f')(i) \to H^n(X, Y, f)(i) \tag{4.2.4.1}$$

induced by morphisms of varieties $h' : X \to X'$ compatible with subvarieties and potentials. The Betti realisation of this morphism is the corresponding morphisms of rapid decay cohomology groups. Let $Z \subseteq Y \subseteq X$ be a pair of closed immersions and $f$ a regular function on $X$. There is a long exact sequence of exponential motives

$$\cdots \longrightarrow H^n(X, Y, f) \longrightarrow H^n(X, Z, f) \longrightarrow H^n(Y, Z, f|_Y) \longrightarrow H^{n+1}(X, Y, f) \longrightarrow \cdots \tag{4.2.4.2}$$

realising to the corresponding long exact sequence in rapid decay cohomology. Indeed, all morphisms in the sequence (4.2.4.2) are moprhisms of motives because they come from morphisms in the quiver $\mathrm{Q}^{\exp}(k)$, and the sequence is also exact because the corresponding sequence of vector spaces is so. Finally, there are isomorphisms

$$H^{n+1}(X \times \mathbb{G}_m, (Y \times \mathbb{G}_m) \cup (X \times \{1\}), f \boxplus 0)(i+1) \to H^n(X, Y, f)(i) \tag{4.2.4.3}$$

in $\mathbf{M}^{\exp}(k)$ realising to the Künneth isomorphisms. Of course, the above are not all morphisms in the category $\mathbf{M}^{\exp}(k)$—taking compositions and linear combinations produces many other morphisms which are not of the elementary shapes (4.2.4.1), (4.2.4.2) or (4.2.4.3).

Lemma 4.2.5. — *For every pair of varieties $Y \subseteq X$ and every regular function $f\colon X \to \mathbb{A}^1$, there is a canonical isomorphism of motives*

$$H^n(X, Y, f) \xrightarrow{\cong} H^{n+1}(X \times \mathbb{A}^1, (Y \times \mathbb{A}^1) \cup \Gamma, p)$$

*where $\Gamma \subseteq X \times \mathbb{A}^1$ is the graph of $f$ and $p\colon X \times \mathbb{A}^1 \to \mathbb{A}^1$ is the projection.*

Proof. This follows essentially from the previous remarks and elements of the proof of Proposition 3.2.2. Associated with the triple $(Y \times \mathbb{A}^1) \subseteq (Y \times \mathbb{A}^1) \cup \Gamma \subseteq (X \times \mathbb{A}^1)$ is a long exact sequence in $\mathbf{M}^{\exp}(k)$. The motives $H^n(X \times \mathbb{A}^1, Y \times \mathbb{A}^1, p)$ appearing in this sequence are zero for all $n$, so the sequence breaks down to isomorphisms

$$H^n((Y \times \mathbb{A}^1) \cup \Gamma, Y \times \mathbb{A}^1, p) \xrightarrow{\cong} H^{n+1}(X \times \mathbb{A}^1, (Y \times \mathbb{A}^1) \cup \Gamma, p). \tag{4.2.5.1}$$

The inclusion $h\colon X \to (Y \times \mathbb{A}^1) \cup \Gamma$ given by $h(x) = (x, f(x))$ sends $Y \subseteq X$ to $Y \times \mathbb{A}^1$ and satisfies $p \circ h = f$, hence a morphism

$$h^*\colon H^n((Y \times \mathbb{A}^1) \cup \Gamma, Y \times \mathbb{A}^1, p) \to H^n(X, Y, f) \tag{4.2.5.2}$$

in $\mathbf{M}^{\exp}(k)$. By excision, this morphism induces an isomorphism in rapid decay cohomology, hence is an isomorphism of motives. The composite of (4.2.5.1) and (4.2.5.2) is what we sought. □

4.2.6. — Let us now show how Nori's universal property is used to construct *realisation functors*. Let $\mathbf{A}$ be an abelian $\mathbb{Q}$-linear category equipped with a faithful exact functor $h\colon \mathbf{A} \to \mathbf{Vec}_{\mathbb{Q}}$, and suppose that we are given a cohomology theory for triples $(X, Y, f)$ with values in $\mathbf{A}$ which is comparable to rapid decay cohomology. Precisely, that means we have a quiver representation

$$\sigma : \mathrm{Q}^{\exp}(k) \to \mathbf{A} \qquad [X, Y, f, n, i] \longmapsto H^n_{\mathbf{A}}(X, Y)(i) \tag{4.2.6.1}$$

and an isomorphism between $h \circ \sigma$ and the Betti representation $\rho$. Nori's universal property as stated in Theorem 4.1.11 applies, yielding a faithful and exact functor

$$\mathrm{R}_{\mathbf{A}}\colon \mathbf{M}^{\exp}(k) \to \mathbf{A} \tag{4.2.6.2}$$

which we call *realisation functor*. A typical examples of such a cohomology theory is the representation associating with $[X, Y, f, n, i]$ the object $H^n_{\mathrm{perv}}(X, Y)(i)$ of $\mathbf{Perv}_0$, in which case we choose for $\mathbf{A}$ the category $\mathbf{Perv}_0$ and for $h$ the fibre near infinity. It can and will happen that we want to study cohomology theories and realisation functors with values in a category which is not $\mathbb{Q}$-linear but $F$-linear for some field of characteristic zero, typically $F = k$ or $F = \mathbb{Q}_\ell$ or $F = \mathbb{C}$. In that case we can not use Theorem 4.1.11 directly, but have to use the following astuce. Suppose we have cohomology theory such as (4.2.6.1) where now $\mathbf{A}$ is $F$-linear with a faithful and exact functor $\mathbf{A} \to \mathbf{Vec}_F$ and a natural isomorphism

$$H^n_{\mathbf{A}}(X, Y)(i) \otimes_F B \cong H^n_{\mathrm{rd}}(X, Y)(i) \otimes_{\mathbb{Q}} B \tag{4.2.6.3}$$

of $B$-vector spaces for some large field $B$ containing $F$. Let $\mathbf{A}^+$ be the category whose objects are triples $(A, V, \alpha)$ consisting of an object $A$ of $\mathbf{A}$, a rational vector space $V$, and an isomorphism of $B$-vector spaces $h(A) \otimes_F B \cong V \otimes_{\mathbb{Q}} B$. The category $\mathbf{A}^+$ is $\mathbb{Q}$-linear, with a faithful and exact functor $h\colon \mathbf{A} \to \mathbf{Vec}_{\mathbb{Q}}$ sending $(A, V, \alpha)$ to $V$. Combining the given representation $\sigma : \mathrm{Q}^{\exp}(k) \to \mathbf{A}$ with (4.2.6.3), we obtain a representation $\sigma^+ : \mathrm{Q}^{\exp}(k) \to \mathbf{A}^+$ such that the equality $h \circ \sigma^+ = \rho$ holds, hence from Nori's universal property a realisation functor $\mathrm{R}_{\mathbf{A}^+} : \mathbf{M}^{\exp}(k) \to \mathbf{A}^+$. We obtain a functor (4.2.6.2) by composing $\mathrm{R}_{\mathbf{A}^+}$ with the forgetful functor $\mathbf{A}^+ \to \mathbf{A}$.

4.2.7. — Much of the strength of Nori's theories of motives, among which we count our category of exponential motives, stems from the fact that there are many variants of the Betti representation $\rho\colon \mathrm{Q}^{\exp}(k) \to \mathbf{Vec}_{\mathbb{Q}}$ which produce the same category of exponential motives.

Proposition 4.2.8. — *Let $\mathrm{Q}^{\exp}_{\mathrm{aff}}(k)$ be the full subquiver of $\mathrm{Q}^{\exp}(k)$ whose objects are those tuples $[X, Y, f, n, i]$ where $X$ is an affine variety. The functor*

$$\langle \mathrm{Q}^{\exp}_{\mathrm{aff}}(k), \rho\rangle \to \langle \mathrm{Q}^{\exp}(k), \rho\rangle = \mathbf{M}^{\exp}(k)$$

*induced by the inclusion* $\mathrm{Q}^{\exp}_{\mathrm{aff}}(k) \to \mathrm{Q}^{\exp}(k)$ *is an equivalence of categories.*

## 4.3. The derived category of exponential motives

In a wide range of contexts, spectral sequences associated to simplicial or filtered spaces are a powerful tool when it comes to computing cohomology. We would like to use these techniques to compute exponential motives. The difficulty in doing so stems from the fact that $H^n(X,Y,f)$ is not defined as the homology in degree $n$ of a complex, as it is the case for most familiar cohomology theories. Our goal in this section is to fabricate adequately functorial complexes which compute exponential motives, as it is done for usual motives in [**Nor**].

Definition 4.3.1. — A triple $[X,Y,f]$ consisiting of a variety $X$ over $k$, a closed subvariety $Y \subseteq X$ and a regular function $f$ is said to be *cellular* in degree $n$ if $H^p_{\mathrm{rd}}(X,Y,f) = 0$ for $p \neq n$.

We write $\mathrm{Q}^{\exp}_{\mathrm{c}}(k)$ for the full subquiver of $\mathrm{Q}^{\exp}(k)$ of those objects $[X,Y,f,n,i]$ such that $X$ is affine of dimension $\leqslant n$ and $[X,Y,f]$ is cellular in degree $n$. We equip $\mathrm{Q}^{\exp}_{\mathrm{c}}(k)$ with the restriction of the Betti representation $\rho$ from 4.2.2, so that the inclusion $\mathrm{Q}^{\exp}_{\mathrm{c}}(k) \subseteq \mathrm{Q}^{\exp}(k)$ can be seen as a morphism of quiver representations. We set $\mathbf{M}^{\exp}_{\mathrm{c}}(k) = \langle \mathrm{Q}^{\exp}_{\mathrm{c}}(k), \rho\rangle$ and call *canonical* the functor

$$\mathbf{M}^{\exp}_{\mathrm{c}}(k) \;\to\; \mathbf{M}^{\exp}(k) \tag{4.3.1.1}$$

induced by the inclusion $\mathrm{Q}^{\exp}_{\mathrm{c}}(k) \to \mathrm{Q}^{\exp}(k)$.

Theorem 4.3.2. — *There exists a quiver representation* $\lambda : \mathrm{Q}^{\exp}(k) \to D^b(\mathbf{M}^{\exp}_{\mathrm{c}}(k))$ *such that the following diagram commutes up to natural isomorphisms:*

$$\begin{array}{ccc} \mathrm{Q}^{\exp}_{\mathrm{c}} & \xrightarrow{\text{can.lift}} & \mathbf{M}^{\exp}_{\mathrm{c}} \\ {\scriptstyle\subseteq}\downarrow & \overset{H^0\circ\lambda}{\nearrow} & \downarrow{\scriptstyle\text{can.}} \\ \mathrm{Q}^{\exp} & \xrightarrow{\text{can.lift}} & \mathbf{M}^{\exp} \end{array}$$

*Moreover, equalities* $\lambda([X,Y,f,n,i]) = \lambda([X,Y,f,0,0])[-n](i)$ *hold and, for all tuples* $[X,Y,Z,f]$, *the triangles*

$$\lambda([X,Y,f,n,i]) \to \lambda([X,Z,f,n,i]) \to \lambda([Y,Z,f|_Y,n,i]) \to \lambda([X,Y,f,n+1,i])$$

*are exact, where morphisms are the images under* $\lambda$ *of the corresponding morphisms of type (a) for inclusions and of type (b) for the triple.*

4.3.3. — The construction of $\lambda$, in particular the construction of the complexes

$$C^\bullet(X,Y,f) = \lambda([X,Y,f,0,0])$$

uses two essential ingredients: One is the Basic Lemma 3.3.3 which we use to define a complex for every object $[X,Y,f,n,i]$ where $X$ is affine, and the other is Jouanoulou's trick, which permits us to replace a general variety with an affine one which is homotopic to it. Having done so, we obtain a complex in $\mathbf{M}_{\mathrm{c}}^{\mathrm{exp}}$ which is our candidate for $\lambda([X,Y,f,n,i])$, but depends on several choices. Once we look at the complex as an object in the derived category $D^b(\mathbf{M}_{\mathrm{c}}^{\mathrm{exp}})$, we get rid of all dependence on these choices.

4.3.4. — Let us recall the following observation, due to Jouanolou [**Jou73**, Lemme 1.5]: For every quasiprojective variety $X$ over $k$, there exists an affine variety $\widetilde{X}$ and a morphism $p\colon \widetilde{X} \to X$ such that each fibre $p^{-1}(x)$ is isomorphic to $\mathbb{A}^d$ for some $d \geqslant 0$ (but there is no such thing as a zero-section $X \to \widetilde{X}$). In particular, the induced continuous map $\widetilde{X}(\mathbb{C}) \to X(\mathbb{C})$ is a homotopy equivalence! The proof is simple: For $X = \mathbb{P}^n$ take for $\widetilde{X}$ the variety of $(n+1)\times(n+1)$ matrices of rank 1 up to scalars with its obvious map to $\mathbb{P}^n$, and for general $X$ choose a projective embedding and do a pullback. Let us call such a morphism $p\colon \widetilde{X} \to X$ an *affine homotopy replacement*.

Jouanoulou's trick does not give a functorial homotopy replacement of varieties $X$ by affine $\widetilde{X}$, but nearly so. Given a morphism of varieties $Y \to X$, we can replace first $X$ with an affine $\widetilde{X} \to X$, and then $Y$ with an affine homotopy replacement $\widetilde{Y}$ of the fibre product $Y \times_X \widetilde{X}$. The map $\widetilde{Y} \to Y$ is an affine homotopy replacement, and we obtain a morphism $\widetilde{Y} \to \widetilde{X}$ which lifts the given morphism $Y \to X$. This procedure can be generalised to the case of several morphisms from $Y \to X$, but not to arbitrary diagrams of varieties.

Definition 4.3.5. — Let $X$ be an affine variety over $k$, let $Z \subseteq Y \subseteq X$ be closed subvarieties and let $f$ be a regular function on $X$. A *cellular filtration* of $[X,Y,Z,f]$ is a chain of closed immersions

$$\varnothing \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X_{p-1} \subseteq X_p \subseteq \cdots \subseteq X_d = X \qquad (4.3.5.1)$$

where each $X_p$ is of dimension $\leqslant p$, such that the triples

$$[X_p, X_{p-1}, f|_{X_p}], \quad [Y_p, Y_{p-1}, f|_{Y_p}], \quad [Z_p, Z_{p-1}, f|_{Z_p}], \quad [X_p, Y_p \cup X_{p-1}, f|_{X_p}], \quad [Y_p, Z_p \cup Y_{p-1}, f|_{Y_p}]$$

are cellular in degree $p$, for $Y_p := X_p \cap Y$ and $Z_p := X_p \cap Z$. By a cellular filtration of $[X,Y,f]$ we understand a cellular filtration of $[X,Y,\varnothing,f]$.

Proposition 4.3.6. — *Let $X$ be an affine variety over $k$, let $Z \subseteq Y \subseteq X$ be closed subvarieties and let $f$ be a regular function on $X$. There exist cellular filtrations of $[X,Y,Z,f]$, and every filtration of $X$ by closed subvarieties $X_p$ of dimension $\leqslant p$ is contained in a cellular filtration.*

Proof. This is a direct consequence of the basic lemma 3.3.3: Suppose we are given a filtration of the form (4.3.5.1), which satisfies the cellularity condition for $j \geqslant p+1$. By 3.3.3 there exists a closed subvariety $Z$ of dimension $\leqslant p-1$ of $X_p$ such that the triples

$$[X_p, X_{p-1}\cup Z, f|_{X_p}], \quad [Y_p, Y_{p-1}\cup(Y_p\cap Z), f|_{Y_p}], \quad [Z_p, Z_{p-1}\cup(Z_p\cap Z), f|_{Z_p}]$$
$$[X_p, (Y_p\cup X_{p-1})\cup Z, f|_{X_p}], \quad [Y_p, (Z_p\cup Y_{p-1})\cup(Z\cap Y_p), f|_{Y_p}]$$

are cellular in degree $p$. Replace then $X_{p-1}$ with $X_{p-1}\cup Z$ and continue by induction on $p$. □

4.3.7. — Let $X$ be an affine variety over $k$, together with a regular function $f$, and let $Y \subseteq X$ be a closed subvariety. Choose a cellular filtration $X_*$ of $[X,Y,f]$ and set $Y_p = X_p \cap Y$. We will consider the complex

$$C^\bullet(X_*, Y_*, f) = \big[\cdots \to H^p(X_p, Y_p \cup X_{p-1}, f|_{X_p}) \xrightarrow{d_p} H^{p+1}(X_{p+1}, Y_{p+1} \cup X_p, f|_{X_{p+1}}) \to \cdots\big] \tag{4.3.7.1}$$

in the category $\mathbf{M}^{\exp}_{\mathrm{c}}(k)$, where the differential $d_p$ is the connecting morphism in the long exact sequence associated with the triple $X_{p-1} \subseteq X_p \subseteq X_{p+1}$ and the sheaf on $X$ which computes the cohomology of the pair $[X, Y]$. By this we mean the following: For every constructible sheaf $F$ on $X$ and every triple $X_{p-1} \subseteq X_p \subseteq X_{p+1}$ there is a short exact sequence of sheaves on $X$

$$0 \to F_{[X_{p+1}, X_{p-1}]} \to F_{[X_p, X_{p-1}]} \to F_{[X_{p+1}, X_p]} \to 0$$

hence a long exact sequence

$$\cdots \to {}^p\mathcal{H}^n(Rf_*F_{[X_{p+1}, X_{p-1}]}) \to {}^p\mathcal{H}^n(Rf_*F_{[X_p, X_{p-1}]}) \xrightarrow{\partial} {}^p\mathcal{H}^{n+1}(Rf_*F_{[X_{p+1}, X_p]}) \to \cdots$$

of perverse sheaves on the affine line. Applying the functor $\Pi$ and taking fibres at infinity, this yields the exact sequence of vector spaces

$$\cdots \to H^n_{\mathrm{rd}}(X_{p+1}, X_{p-1}, f; F) \to H^n_{\mathrm{rd}}(X_p, X_{p-1}, f; F) \xrightarrow{\partial} H^{n+1}_{\mathrm{rd}}(X_{p+1}, X_p, f; F) \to \cdots \tag{4.3.7.2}$$

by definition of rapid decay cohomology with coefficients in a constructible sheaf. We consider (4.3.7.2) for the terms of the standard short exact sequence of sheaves on $X$

$$0 \to \underline{\mathbb{Q}}_{[X,Y]} \to \underline{\mathbb{Q}}_X \to \underline{\mathbb{Q}}_Y \to 0.$$

Taking into account that

$$\begin{aligned} H^n_{\mathrm{rd}}(X_p, X_{p-1}, f; \underline{\mathbb{Q}}_{[X,Y]}) &= H^n_{\mathrm{rd}}(X_p, Y_p \cup X_{p-1}, f|_{X_p}) \\ H^n_{\mathrm{rd}}(X_p, X_{p-1}, f; \underline{\mathbb{Q}}_X) &= H^n_{\mathrm{rd}}(X_p, X_{p-1}, f|_{X_p}) \\ H^n_{\mathrm{rd}}(X_p, X_{p-1}, f; \underline{\mathbb{Q}}_Y) &= H^n_{\mathrm{rd}}(Y_p, Y_{p-1}, f|_{Y_p}), \end{aligned}$$

(4.3.7.2) and the cellularity assumptions yield a morphism of short exact sequences of vector spaces

$$\begin{array}{ccccccccc} 0 \to & H^p_{\mathrm{rd}}(X_p, Y_p \cup X_{p-1}, f_{|X_p}) & \longrightarrow & H^p_{\mathrm{rd}}(X_p, X_{p-1}, f_{|X_p}) & \longrightarrow & H^p_{\mathrm{rd}}(Y_p, Y_{p-1}, f_{|Y_p}) & \to 0 \\ & \downarrow{\scriptstyle d_p} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & \\ 0 \to & H^{p+1}_{\mathrm{rd}}(X_{p+1}, Y_{p+1} \cup X_p, f_{|X_{p+1}}) & \to & H^{p+1}_{\mathrm{rd}}(X_{p+1}, X_p, f_{|X_{p+1}}) & \longrightarrow & H^{p+1}_{\mathrm{rd}}(Y_{p+1}, Y_p, f_{|Y_{p+1}}) & \to 0 \end{array}$$

in which the differential of (4.3.7.1) appears. All vector spaces in this diagram underly objects of $\mathbf{M}^{\exp}_{\mathrm{c}}(k)$. This diagram shows as well that $d_p$ is a morphism in $\mathbf{M}^{\exp}_{\mathrm{c}}(k)$ rather than just a morphism of vector spaces, indeed, all other morphisms in the diagram are morphisms in $\mathbf{M}^{\exp}_{\mathrm{c}}(k)$ since they either are given by inclusions of pairs or by connecting morphisms of triples, and hence come from morphisms in $\mathrm{Q}^{\exp}_{\mathrm{c}}(k)$. That the composite $d_{p-1} \circ d_p$ is zero follows from the fact that for any chain of closed subvarieties $X_{p-2} \subseteq X_{p-1} \subseteq X_p \subseteq X_{p+1}$ of $X$ and any sheaf $F$ on $X$, the composite

$$H^{p-1}_{\mathrm{rd}}([X_{p-1}, X_{p-2}], F) \to H^p_{\mathrm{rd}}([X_p, X_{p-1}], F) \to H^{p+1}_{\mathrm{rd}}([X_{p+1}, X_p], F)$$

is zero. The complex $C^*(X_*, Y_*, f)$ is functorial in the obvious way for morphisms of filtered pairs: Let $h\colon X' \to X$ be a morphism of affine varieties over $k$, restricting to a morphism $Y' \to Y$

between closed subvarieties, set $f' := f \circ h$, and let $X_*$ and $X'_*$ be cellular filtrations for $[X, Y, f]$ and $[X', Y', f']$ such that $h(X'_p)$ is contained in $X_p$ and $h(Y'_p)$ in $Y_p$ for all $p \geqslant 0$. The morphism

$$C^*(h) : C^*(X_*, Y_*, f) \to C^*(X'_*, Y'_*, f') \tag{4.3.7.3}$$

shall be the one induced by the morphism $H^p(X_p, Y_p \cup X_{p-1}, f|_{X_p}) \to H^p(X'_p, Y'_p \cup X'_{p-1}, f'|_{X'_p})$ given by the restriction of $h$ to $X'_p$.

4.3.8. — We now turn to the proof that the cohomology of the complex $C^\bullet(X_*, Y_*, f)$ computes the exponential motives $H^n(X, Y, f)$. Recall from (3.2.4.1) that $\Gamma_f \colon \mathrm{Sh}(X) \to \mathrm{Vec}_{\mathbb{Q}}$ is the left exact functor obtained by composing in that order: the direct image functor $f_*$, taking the tensor product $- \boxtimes j_! \underline{\mathbb{Q}}_{\mathbb{G}_m}$ on $\mathbb{A}^2$, the direct image functor $\mathrm{sum}_*$ and the fibre functor $\Psi_\infty$.

Lemma 4.3.9. — *Let $X$ be an affine variety over $k$, together with a regular function $f$, and let $Y \subseteq X$ be a closed subvariety. Choose a cellular filtration $X_*$ of $[X, Y, f]$. There is a natural isomorphism in the derived category of vector spaces*

$$C^\bullet(X_*, Y_*, f) \cong R\Gamma_f(\underline{\mathbb{Q}}_{[X,Y]}). \tag{4.3.9.1}$$

Proof. That the complex $R\Gamma_f(\underline{\mathbb{Q}}_{[X,Y]})$ computes rapid decay cohomology was explained in Proposition 3.2.5. The complex on the right hand is calculated by choosing an injective resolution

$$I_* = [I_0 \to I_1 \to I_2 \to \cdots]$$

of the sheaf $\underline{\mathbb{Q}}_{[X,Y]}$ and applying to this resolution the functor $\Gamma_f$. On the left hand side we have a complex of motives, which has an underlying complex of vector spaces. It is given in degree $p$ by the vector space $H^p(X_p, X_{p-1} \cup Y_p, f)$ which is the same as $R\Gamma_f(\underline{\mathbb{Q}}_{[X_p, X_{p-1} \cup Y_p]})$ by Proposition 3.2.5. Thus, the claim of the Lemma is the following:

**Claim:** Let $F$ be a sheaf on $X$, and let $X_*$ be a finite exhaustive filtration of $X$ by closed subspaces $X_p$ such that $H^n(X_p, X_{p-1}, f|_{X_p}, F)$ is zero for $n \neq p$. Then the complex of vector spaces

$$\cdots \to H^{p-1}(X_{p-1}, X_{p-2}, f, F) \to H^p(X_p, X_{p-1}, f, F) \to H^{p+1}(X_{p+1}, X_p, f, F) \to \cdots \tag{4.3.9.2}$$

is isomorphic to $R\Gamma_f(F)$ in the derived category of vector spaces.

To see this, choose an injective resolution $F \to I_*$ of $F$. The long exact sequence (4.3.9.2) is natural in $F$, so if we apply it to $I_*$ we obtain a double complex, hence a spectral sequence

$$E_1^{p,q} = H^{p+q}(X_p, X_{p-1}, f; F) \Longrightarrow H^{p+q}(X, f, F).$$

By the assumption that the filtration is cellular, the term $E_1^{p,q}$ vanishes for $q \neq 0$, hence the spectral sequence degenerates at the second page, yielding the desired quasi-isomorphism of complexes.

Naturality of the isomorphism (4.3.9.1) for morphisms of filtered pairs follows from naturality of (4.3.9.2) in $X_*$ and $F$. □

Proposition 4.3.10. — *Let $X$ be an affine variety over $k$, together with a regular function $f$, and let $Y \subseteq X$ be a closed subvariety. Choose a cellular filtration $X_*$ of $[X,Y,f]$. There is a canonical isomorphism in $\mathbf{M}^{\exp}(k)$*

$$H^p(C^*(X_*,Y_*,f)) \cong H^p(X,Y,f)$$

*which is natural for morphisms of filtered pairs. If $X$ is of dimension $\leqslant n$ and $[X,Y,f]$ is cellular in degree $n$, then this isomorphism is an isomorphism in $\mathbf{M}^{\exp}_{\mathrm{c}}(k)$.*

Proof. The cohomology of $C^*(X_*,Y_*,f)$ in degree $p$ is the object

$$H^p(C^*(X_*,Y_*,f)) = \frac{\ker\big(H^p(X_p, Y_p \cup X_{p-1}, f_p) \to H^{p+1}(X_{p+1}, Y_{p+1}\cup X_p, f_{p+1})\big)}{\operatorname{im}\big(H^{p-1}(X_{p-1}, Y_{p-1}\cup X_{p-2}, f_{p-1}) \to H^p(X_p, Y_p \cup X_{p-1}, f_p)\big)} \qquad (4.3.10.1)$$

in $\mathbf{M}^{\exp}_{\mathrm{c}}(k)$ and we wish to show that this object is naturally isomorphic to $H^p(X,Y,f)$ in $\mathbf{M}^{\exp}(k)$, and even in $\mathbf{M}^{\exp(k)}_{\mathrm{c}}$ whenever $[X,Y,f]$ is cellular. To treat cases uniformly, pick any finite subquiver $Q$ of $\mathrm{Q}^{\exp}(k)$ or of $\mathrm{Q}^{\exp}_{\mathrm{c}}(k)$ which contains at least $[X,Y,f,p,0]$, the $[X_p, Y_p\cup X_{p-1}, f_p, p, 0]$, the morphisms coming from inclusions, and the connecting morphisms of triples, subject to future enlargment. Set $E = \operatorname{End}(\rho|_Q)$. For all integers $q < p$ and $n$, we have $H^n(X_p, Y_p \cup X_q, f_p) = 0$ unless $q < n \leqslant p$. Indeed, this is true by definition if $q = p-1$, and follows in general by induction on $p - q$ using the long exact sequence associated with the triple $X_q \subseteq X_{p-1} \subseteq X_p$. This explains why the morphisms

$$H^p(X,Y,f) \to H^p(X_{p+1}, Y_{p+1}, f_{p+1}) \longleftarrow H^p(X_{p+1}, Y_{p+1}\cup X_{p-2}, f_{p+1}) \qquad (4.3.10.2)$$

are isomorphisms of vector spaces, and also explains the surjections and injections in the following diagram, whose exact rows and columns are pieces of the long exact sequences associated with triples out of the quadruple $X_{p-2} \subseteq \cdots \subseteq X_{p+1}$.

$$\begin{array}{ccccc}
H^{p-1}(X_{p-1}, Y_{p-1}\cup X_{p-2}, f_{p-1}) & \xrightarrow{\partial} & H^p(X_{p+1}, Y_{p+1}\cup X_{p-1}, f_{p+1}) & \twoheadrightarrow & H^p(X_{p+1}, Y_{p+1}\cup X_{p-2}, f_{p+1}) \\
{*}\Big\| & & \Big\downarrow & & \Big\downarrow \\
H^{p-1}(X_{p-1}, Y_{p-1}\cup X_{p-2}, f_{p-1}) & \xrightarrow[*]{\partial} & H^p(X_p, Y_p \cup X_{p-1}, f_p) & \twoheadrightarrow & H^p(X_p, Y_p \cup X_{p-2}, f_p) \\
 & & {*}\Big\downarrow\partial & & \partial\Big\downarrow \\
 & & H^{p+1}(X_{p+1}, Y_{p+1}\cup X_p, f_{p+1}) & \overset{}{\underset{*}{=\!=}} & H^{p+1}(X_{p+1}, Y_{p+1}\cup X_p, f_{p+1})
\end{array}$$

This diagram is a diagram of vector spaces where all morphisms labelled with a $*$ are morphisms of $E$-modules between $E$-modules. But then the whole diagram is a diagram of $E$-modules, in only one possible way. Now we have an $E$-module structure on $H^p(X,Y,f)$ and on $H^p(X_{p+1}, Y_{p+1}\cup X_{p-2}, f_{p+1})$, and we need to show that the isomorphisms (4.3.10.2) are isomorphisms of $E$-modules after possibly enlarging $Q$. In the case where we work with subquivers of $\mathrm{Q}(k)$ we add to $Q$ the two morphisms of pairs needed to define (4.3.10.2) and are done. If we work with cellular pairs only, then $X$ has dimension $\leqslant p$ and $[X,Y]$ is cellular in degree $p$, and we enlarge $Q$ as follows: By the Basic Lemma 3.3.3, there exists a closed $Z \subseteq X$ of dimension $\leqslant p-1$ such that $H^p(X,Y',f)$ is cellular in degree $p$ for $Y' := Y \cup X_{p-1} \cup Z$. Add the morphism $[X,Y',f,n,0] \to [X,Y,f,n,0]$ to $Q$ so that $H^p(X,Y',f) \to H^p(X,Y,f)$ is an $E$-linear morphism. It is surjective for dimension reasons, and the diagram of $E$-modules and linear maps

$$\begin{array}{ccc} H^p(X,Y',f) & \xrightarrow{\;*\;} & H^p(X,Y,f) \\ {\scriptstyle *}\downarrow & & \cong\downarrow{\scriptstyle u} \\ H^p(X_p,Y_p\cup X_{p-1},f_p) & \xrightarrow[*]{} & H^p(X_p,Y_p\cup X_{p-2},f_p) \end{array}$$

commutes, where the isomorphism $u$ is induced by (4.3.10.2). All morphisms labelled $*$ are $E$-linear hence so is $u$. Altogether, we conclude that the homology in the middle of

$$H^{p-1}(X_{p-1},Y_{p-1}\cup X_{p-2},f_{p-1}) \to H^p(X_p,Y_p\cup X_{p-1},f_p) \to H^{p+1}(X_{p+1},Y_{p+1}\cup X_p,f_{p+1})$$

is indeed canonically isomorphic to $H^p(X,Y,f)$ as an $E$-module, which is what we had to show. Naturality of the isomorphism for morphisms of filtered pairs follows from functoriality of (4.3.10.2). □

Corollary 4.3.11. — *Let $X$ and $X'$ be affine varieties and $h\colon [X,Y,f]\to[X',Y',f']$ be a morphism in $Q^{\exp}(k)$. Let $X_*$ and $X'_*$ be cellular filtrations of $[X_1,Y',f']$ and $[X',Y',f']$. If $h$ induces an isomorphism in rapid decay cohomology, then the morphism of complexes $C^*([X_*,Y_*,f])\to C^*([X'_*,Y'_*,f])$ defined in (4.3.7.3) is a quasiisomorphism.*

Proof. This follows from the conservativity of the forgetful functor $\mathbf{M}^{\exp}(k)\to\mathbf{Vec}_{\mathbb{Q}}$ and Proposition 4.3.10. □

Proposition 4.3.12. — *Let $X$ be an affine variety over $k$, let $f$ be a regular function on $X$ and let $Z\subseteq Y\subseteq X$ and $Z\subseteq Y$ be closed subvarieties. Let $X_*$ be a cellular filtration of $[X,Y,Z,f]$. The sequence of complexes with morphisms given by (4.3.7.3) for inclusions*

$$0\to C^*([X_*,Y_*])\to C^*([X_*,Z_*])\to C^*([Y_*,Z_*])\to 0 \tag{4.3.12.1}$$

*is degreewise exact.*

Proof. The sequences in question are sequences in $\mathbf{M}^{\exp}_{\mathrm{c}}(k)$, but in order to show that they are exact it suffices to show that the underlying sequence in **Vec** are exact. But that immediately follows from the definition of cellular filtrations and a diagram chase. □

We have now all the ingredients needed for the proof of the main result of this section.

Proof of Theorem 4.3.2. For each object $[X,Y,f,n,i]$ of the quiver $\mathrm{Q}^{\exp}(k)$ and each cellular filtration $X_*$ of $[X,Y,f]$, we consider the complex $C^*(X_*,Y_*,f)[-n](i)$ obtained from (4.3.7.1) by shifting and twisting degree by degree. Let us define $\lambda$ as follows:

$$\lambda([X,Y,f,n,i]) = \operatorname*{colim}_{\widetilde{X}\to X}\lim_{\widetilde{X}_*} C^*(\widetilde{X}_*,\widetilde{Y}_*,\widetilde{f})[-n](i)$$

on objects $[X,Y,f,n,i]$ of $\mathrm{Q}^{\exp}(k)$, where the limit runs over all cellular filtrations of the triple $[\widetilde{X},\widetilde{Y},f]$ and the colimit over all affine homotopy replacements $\Pi\colon\widetilde{X}\to X$, setting $\widetilde{Y}=\widetilde{X}\times_X Y$ and $\widetilde{f}=f\circ p$. These colimits and limits exist in the derived category $D^b(\mathbf{M}^{\exp}_{\mathrm{c}}(k))$. Indeed, all transition maps are isomorphisms by Corollary 4.3.11. From the practical point of view, $\lambda([X, Y, f, n, i])$ is isomorphic to any of the complexes $C^*(\widetilde{X}_*, \widetilde{Y}_*, \widetilde{f})[-n](i)$ up to a unique isomorphism in $D^b(\mathbf{M}_{\mathrm{c}}^{\exp}(k))$, and the use of the limiting processes is only an artifact to get rid of choices[2]. We define $\lambda$ on morphisms as follows:

Type (a): Let $h\colon [X, Y, f, n, i] \to [X', Y', f', n, i]$ be given by a morphism of varieties $h\colon X' \to X$ such that $h(Y') \subseteq Y$ and $f' = f \circ h$. From (4.3.7.3) we obtain a morphism

$$C^\bullet(h)\colon C^\bullet(\widetilde{X}'_*, \widetilde{Y}'_*, f') \longrightarrow C^\bullet(\widetilde{X}_*, \widetilde{Y}_*, f)$$

for suitable affine homotopy replacements and cellular filtrations, and set $\lambda(f) = C^\bullet(f)[-n](i)$.

Type (b): Let $d : [Y, Z, n, i] \to [X, Y, n+1, i]$ be given by closed immersions $Z \subseteq Y \subseteq X$ between affine varieties. Choose an affine homotopy replacement $\widetilde{X} \to X$, set $\widetilde{Y} = \widetilde{X} \times_X Y$ and $\widetilde{Z} = \widetilde{X} \times_X Z$ and cellular filtration of the triple $[\widetilde{X}, \widetilde{Y}, \widetilde{Z}]$. From Proposition 4.3.12 we obtain an degreewise exact sequence of complexes

$$0 \to C^*([\widetilde{X}_*, \widetilde{Y}_*]) \xrightarrow{\ r\ } C^*([\widetilde{X}_*, \widetilde{Z}_*]) \xrightarrow{\ s\ } C^*([\widetilde{Y}_*, \widetilde{Z}_*] \to 0 \tag{4.3.12.2}$$

where $r$ and $s$ induced by inclusions, hence a morphism in $D^b(\mathbf{M}_{\mathrm{c}})$ given by the hat

$$\begin{array}{ccc} & \mathrm{Cone}(r) & \\ {\scriptstyle \simeq \text{ induced by } s}\swarrow & & \searrow \\ C^*([Y,Z]) & \overset{\partial}{\dashrightarrow} & C^*([X,Y])[-1] \end{array}$$

and define $\lambda(d) = \partial[-n](i)$.

Type (c): If $\widetilde{X} \to X$ is an affine homotopy replacement, then so is $\widetilde{X} \times \mathbb{G}_m \to X \times \mathbb{G}_m$. If $\widetilde{X}_*$ a cellular filtration of $[\widetilde{X}, \widetilde{Y}]$, then the $\widetilde{X}_p \times \mathbb{G}_m \subseteq \widetilde{X} \times \mathbb{G}_m$ form a cellular filtration of $[\widetilde{X} \times \mathbb{G}_m, \widetilde{Y} \times \mathbb{G}_m \cup \widetilde{X} \times \{1\}]$. Hence there is a canonical isomorphism of complexes

$$C^*([\widetilde{X}_* \times \mathbb{G}_m, \widetilde{Y}_* \times \mathbb{G}_m \cup \widetilde{X}_* \times \{1\}])(1) \to C^*([\widetilde{X}_*, \widetilde{Y}_*]) \tag{4.3.12.3}$$

obtained from the corresponding isomorphisms degree-by-degree, and we declare this morphism shifted and twisted by $[-n](i)$ to be the image under $\lambda$ of the morphism of type (c) with target $[X, Y, n, i]$.

Now that we have defined $\lambda$, it remains to show that the diagram in the statement of Theorem 4.3.2 indeed commutes up to natural isomorphisms. All other statement hold by construction. The isomorphisms we seek

$$\lambda([X, Y, n, i]) \cong H^n([X, Y])(i)$$

are those of Proposition 4.3.10 with a twist. Naturality of these isomorphisms for morphisms in $\mathrm{Q}(k)$ is a question on the level of modules, and follows from the fact that the isomorphisms in Proposition 4.3.10 are induced, as morphisms of modules, by the isomorphisms of complexes of Lemma 4.3.9. $\square$

[2] provided a concrete construction of limits in $D^b(\mathbf{M}_{\mathrm{c}}^{\exp}(k))$ is at disposal.

COROLLARY 4.3.13. — *Each object in* $\mathbf{M}^{\exp}(k)$ *is a subquotient of a sum of objects of the form* $H^n(X, Y, f)(i)$, *where* $X = \overline{X} \setminus Y_\infty$ *and* $Y = Y_0 \setminus (Y_0 \cap Y_\infty)$ *for a smooth projective variety* $\overline{X}$ *of dimension n and two normal crossing divisors* $Y_0$ *and* $Y_\infty$ *such that the union* $Y_0 \cup Y_\infty$ *has normal crossings as well.*

PROOF. The combination of Theorem 4.3.2 and Proposition 4.1.7 yields that every object of $\mathbf{M}^{\exp}(k)$ is a subquotient of a sum of exponential motives $M = H^n(X, Y, f)$, where $X$ is an affine variety of dimension $n$ and $Y \subseteq X$ a closed subvariety of dimension $\leqslant n-1$ such that the triple $[X, Y, f]$ is cellular in degree $n$. We are thus reduced to prove the statement for those $M$.

Let $Y' \subseteq X$ be a closed subvariety of dimension $\leqslant n-1$ containing $Y$ and the singular locus of $X$. Since $H^n(Y', Y, f_{|Y'}) = 0$ by Artin vanishing, the long exact sequence (4.2.4.2) shows that the morphism $H^n(X, Y', f) \to M$ is surjective. Up to replacing $Y$ by $Y'$, we may therefore assume that $U = X \setminus Y$ is smooth. Let $\overline{X}$ be a compactification of $X$, $Y_\infty = \overline{X} \setminus X$, and $Y_0$ the closure of $Y$ in $\overline{X}$. Using resolution of singularities,

$$\overline{X}$$

□

## 4.4. Tensor products

In this section, we introduce a tensor product on the category of exponential motives, following Nori's ideas. We shall prove later that with this tensor product structure, $\mathbf{M}^{\exp}(k)$ is a neutral tannakian category, with $\mathrm{R_B}$ as fibre functor.

THEOREM 4.4.1. — *The category* $\mathbf{M}^{\exp}(k)$ *admits a unique* $\mathbb{Q}$*-linear monoidal structure that satisfies the following properties.*

*(1) The forgetful functor* $\mathrm{R_B}\colon \mathbf{M}^{\exp}(k) \to \mathbf{Vec}_{\mathbb{Q}}$ *is strictly monoidal.*
*(2) Künneth morphisms are morphisms of motives.*

*With respect to this monoidal structure and* $R_B$ *as fibre functor,* $\mathbf{M}^{\exp}(k)$ *is a neutral tannakian category.*

By a symmetric monoidal structure on $\mathbf{M}^{\exp}$ we understand a functor

$$\otimes\colon \mathbf{M}^{\exp}(k) \times \mathbf{M}^{\exp}(k) \to \mathbf{M}^{\exp}(k)$$

which we call *tensor product* together with isomorphisms of functors expressing associativity and commutativity of the tensor product, and the properties of $\mathbb{Q}(0) = H^0(\operatorname{Spec} k)$ playing the role of a neutral object. That the forgetful functor or Betti realisation $\mathrm{R_B}\colon \mathbf{M}^{\exp}(k) \to \mathbf{Vec}_{\mathbb{Q}}$ is strictly monoidal means that there exist natural isomorphisms

$$\mathrm{R_B}(M \otimes N) \cong \mathrm{R_B}(M) \otimes \mathrm{R_B}(N) \tag{4.4.1.1}$$

that are compatible with the associativity and commutativity constraints. These isomorphisms will come directly from the construction of the tensor product, and will be equalities. Given relative varieties with functions $[X, Y, f]$ and $[X', Y', f']$, the Künneth morphisms are maps of vector spaces

$$H^n(X, Y, f) \otimes H^{n'}(X', Y', f') \to H^{n+n'}(X \times X', (Y \times X') \cup (X \times Y'), f \boxplus f')$$

and property (2) states that these morphisms are compatible with the motivic structures.

What remains to be done to get the tannakian structure on $\mathbf{M}^{\exp}(k)$ is to prove that, for each object $M$, the functor $- \otimes M$ has a natural right adjoint, which we denote by $\underline{\mathrm{Hom}}(M, -)$, so that the usual adjunction formula holds:

$$\mathrm{Hom}(X \otimes M, Y) \cong \mathrm{Hom}(X, \underline{\mathrm{Hom}}(M, Y)).$$

4.4.2 (Construction of the tensor product). — To ease notations, let us denote by $\mathrm{Q}_c = \mathrm{Q}_c^{\exp}(k)$ the cellular quiver from Definition 4.3.1. We consider the quiver morphism

$$\mathrm{prod}\colon \mathrm{Q}_a \boxtimes \mathrm{Q}_a \to \mathrm{Q}_a$$

given on objects by

$$\mathrm{prod}([X, Y, f, n, i] \boxtimes [X', Y', f', n, i']) = [X \times X', Y \times X' \cup X \times Y', f \boxplus f', n + n', i + i'],$$

and with the evident definition on morphisms. The Künneth formula provides a natural isomorphism

$$H^n(X, Y, f)(i) \otimes H^{n'}(X', Y', f')(i') \cong H^{n+n'}(X \times X', Y \times X' \cup X \times Y', f \boxplus f')(i + i')$$

since all other terms in the Künneth formula are zero for dimension reasons. We obtain hence a morphism of quiver representations

$$\begin{array}{ccc} \mathrm{Q}_c \boxtimes \mathrm{Q}_c & \xrightarrow{\ \mathrm{prod}\ } & \mathrm{Q}_c \\ & {\scriptstyle \rho\boxtimes\rho} \searrow \quad \swarrow {\scriptstyle \rho} & \\ & \mathbf{Vec}_{\mathbb{Q}} & \end{array}$$

which induces a functor

$$\mathbf{M}^{\exp}(k) \times \mathbf{M}^{\exp}(k) \longrightarrow \mathbf{M}^{\exp}(k)$$

4.4.3 (Construction of the commutativity constraint). —

## 4.5. Intermezzo: Simplicial spaces and hypercoverings

## 4.6. Motives of simplicial varieties

Let $U$ and $V$ be open subvarieties of a variety $X$, such that $X$ is the union of $U$ and $V$, and let $f$ be a regular function on $X$. The rapid decay cohomology groups of $X$, $U$, $V$ and $U \cap V$ with respect to the function $f$ are related by a *Mayer–Vietoris* sequence:

$$\cdots \to H^n(X,f) \to H^n(U,f|_U) \oplus H^n(V,f|_V) \to H^n(U\cap V, f_{U\cap V}) \xrightarrow{\partial} H^{n+1}(X,f) \to \cdots$$

All terms in this sequence are motives, and differentials which connect cohomology groups of the same degree are morphisms of motives, since they are induced by inclusions of varieties. Also the connecting morphism $\partial$ is a morphism of motives, but this is not clear a priori, and we shall prove it in this section. As an easy consequence, we establish a projective bundle formula and a sphere bundle formula for exponential motives.

4.6.1. —

4.6.2. — A sheaf on a simplicial topological space $X_\bullet$ is the data of a sheaf $F_n$ on each $X_n$ and compatibility morphisms for faces and cofaces. One can regard sheaves $F_\bullet$ on $X_\bullet$ as sheaves on some convenient site, and using that point of view, the cohomology $H^n(X_\bullet, F_\bullet)$ is defined. There is a spectral sequence of groups

$$E_1^{p,q} = H^q(X_p, F_p) \implies H^{p+q}(X_\bullet, F_\bullet) \tag{4.6.2.1}$$

constructed as follows: An injective resolution of $F_\bullet$ gives an injective resolution of each $F_n$. Applying global sections gives a simplicial complex of vector spaces, hence a double complex. Its horizontal differentials (say) are alternating sums of face maps, and its vertical differentials are induced by the differentials of the resolution. The spectral sequence (4.6.2.1) is the one associated with this double complex for the filtration by columns, see e.g. definition 5.6.1 in [**Wei94**].

Let $\varepsilon\colon X_0 \to X$ be a continuous map between topological spaces coequalising the two face maps $X_1 \to X_0$. Such a map $\varepsilon$ is traditionally called *augmentation*, and we may view it as a morphism from $X_\bullet$ to the constant simplicial space $X$. There is a pair of adjoint functors $(\varepsilon^*, \varepsilon_*)$ between the categories of sheaves on $X_\bullet$ and on $X$. The map $\varepsilon$ is said to have the property of *cohomological descent* if the adjunction transform

$$F \to R\varepsilon_*(\varepsilon^* F)$$

is an isomorphism in the derived category of the category of sheaves on $X$, for every sheaf or complex of sheaves $F$ on $X$. In that case, there is an isomorphism $H^n(X_\bullet, \varepsilon^* F) \to H^n(X, F)$ for every sheaf $F$ on $X$, and the spectral sequence (4.6.2.1) translates to a spectral sequence

$$E_1^{p,q} = H^q(X_p, \varepsilon^* F_p) \implies H^{p+q}(X, F) \tag{4.6.2.2}$$

where the induced filtration on the cohomology groups $H^n(X, F)$ is the one that comes from truncations of $X_\bullet$ (the $q$-skeletons of $X_\bullet$ for $q = 0, 1, 2, \ldots$).

4.6.3. — Let $\varepsilon\colon X_\bullet \to X$ be an augmented simplicial topological space, and let $f\colon X \to \mathbb{R}$ be a continuous function. Set $f_\bullet = f \circ \varepsilon$. For every real $r$, the sets $S_n := \{x \in X_n \mid f_n(x) \geqslant r\}$ form a closed simplicial subspace $S_\bullet$ of $X_\bullet$, augmented to $S := \{x \in X \mid f(x) \geqslant r\}$.

Proposition 4.6.4. — *Let* $(X_\bullet, Y_\bullet, f_\bullet)$ *be a simplicial object as above. A morphism*

$$\varepsilon\colon (X_\bullet, Y_\bullet, f_\bullet) \to (X, Y, f)$$

*satisfies cohomological descent if the morphism* $(X_\bullet, Y_\bullet) \to (X, Y)$ *does.*

Proof. This is a purely topological question. □

4.6.5. — Recall here what a hypercovering is.

Theorem 4.6.6. — *The spectral sequence of a hypercovering is motivic.*

Corollary 4.6.7. — *exponential Mayer–Vietoris*

Proposition 4.6.8. — *Set* $\mathbb{Q}(i) = H^0(\operatorname{Spec} k)(i)$. *There are canonical isomorphisms of motives:*

(1) $H^n(\mathbb{P}^d_k) = \begin{cases} \mathbb{Q}(-i) & n = 2i \leqslant 2d \\ 0 & n \text{ odd or } n > 2d. \end{cases}$

(2) $H^n(\mathbb{A}^d_k \setminus \{0\}) = \begin{cases} \mathbb{Q}(0) & n = 0 \\ \mathbb{Q}(-d) & n = 2d-1 \\ 0 & \text{else.} \end{cases}$

(3) $H^{2d}(X) = \mathbb{Q}(-d)$ *for every projective and geometrically connected variety* $X$ *of dimension* $d$ *over* $k$.

Proof. The morphisms of type (c) in the standard quiver representation of $\mathrm{Q}^{\exp}(k)$ induce isomorphisms $H^d(\mathbb{G}^d_m)(0) \cong H^{d-1}(\mathbb{G}^{d-1}_m)(-1) \cong \cdots \cong H^0(\operatorname{Spec} k)(-d) = \mathbb{Q}(-d)$. From the standard covering of the projective space $\mathbb{P}^d_k$ by $d+1$ affine spaces we obtain a spectral sequence (the Mayer–Vietoris sequence if $d = 1$), in which the connecting morphism

$$H^d(\mathbb{G}^d_m) \to H^{2d}(\mathbb{P}^d_k)$$

appears. This connecting morphism is an isomorphism of vector spaces because $\mathbb{A}^d_k$ is contractible as well as a morphism of motives by Theorem 4.6.6 (or Corollary 4.6.7 for $d = 1$), hence it is an isomorphism of motives. The first statement of the proposition follows from this isomorphism, induction on $d$ and the fact that the inclusion of a hyperplane $\mathbb{P}^{d-1}_k \to \mathbb{P}^d_k$ induces an isomorphism on cohomology in degrees up to $2d-2$. The argument for the second statement is similar—one covers $\mathbb{A}^d_k \setminus \{0\}$ by the affine open $\mathbb{A}^a \times \mathbb{G}_m \times \mathbb{A}^b$ with $a+b = d-1$. To show the third statement of the proposition, choose a projective embedding $f\colon X \to \mathbb{P}^N_k$. The morphism $f$ induces a morphism of motives $\mathbb{Q}(-d) \cong H^{2d}(\mathbb{P}^N_k) \to H^{2d}(X)$ which is an isomorphism of modules, hence an isomorphism of motives. □

Proposition 4.6.9 (Projective bundle formula). — *Let* $X$ *be a variety over* $k$ *equipped with a regular function* $f\colon X \to \mathbb{A}^1$, *let* $Y \subseteq X$ *be a closed subvariety and let* $E \to X$ *be a vector bundle* *of rank $r$ over $X$, with projectivisation $\pi\colon \mathbb{P}(E) \to X$. There is an isomorphism in $\mathbf{M}^{\exp}(k)$ as follows:*

$$H^n(\mathbb{P}(E), \mathbb{P}(E)|_Y, f \circ \pi) \cong \bigoplus_{i=0}^{r-1} H^{n-2i}(X, Y, f)(i).$$

Proof. Proposition 4.6.8 and the Künneth formula settle the case where $E$ is constant. The Künneth isomorphism is indeed a morphism of motives by Theorem 4.4.1. For the general case, choose a finite covering of $X$ by open subvarieties $(U_i)_{i\in I}$ on which $E$ is isomorphic to the constant bundle of rank $r$. With the open covering $(\mathbb{P}(E) \times_X U_i)_{i\in I}$ of $\mathbb{P}(E)$ is associated a simplicial variety, which in turn yields a double complex of motives. The motive $H^*(\mathbb{P}(E), \mathbb{P}(E)|_Y, f \circ \pi)$ is the homology of the associated simple complex as we have explained. We can compute the motive of $\mathbb{P}(\mathbb{A}^r) \times X$ marked in $\mathbb{P}(\mathbb{A}^r) \times Y$ in the same way. For every finite intersection $V$ of the $U_i$'s, choose an isomorphism of bundles $\alpha_V\colon \mathbb{A}^r \times V \cong E \times_X V$. These maps induce isomorphisms of motives

$$H^n(\mathbb{P}(E) \times_X V, \mathbb{P}(E) \times_X (V \cap Y), f \circ \pi) \cong H^n(\mathbb{P}(\mathbb{A}^r) \times V, \mathbb{P}(\mathbb{A}^r) \times (V \cap Y), f \circ \mathrm{proj})$$

which are independent of the choice of $\alpha_V$ because $\mathrm{GL}_r(\mathbb{C})$ is connected. It follows that these isomorphisms commute with the differentials in the double complexes, hence induce an isomorphism of motives. □

Corollary 4.6.10. — *Let $X$ be a variety over $k$, let $f\colon X \to \mathbb{A}^1$ be a regular function and let $\pi\colon E \to X$ be a vector bundle of rank $r$ over $X$. Let $Y \subseteq X$ be a closed subvariety. There is an isomorphism in $\mathbf{M}^{\exp}(k)$ as follows:*

$$H^n(\mathbb{P}(E \oplus \mathcal{O}_X), \mathbb{P}(E) \cup \mathbb{P}(E \oplus \mathcal{O}_X)|_Y, f \circ \pi) = H^{n-2r}(X, Y, f)(-r).$$

Proof. The projective bundle formula established in Proposition 4.6.9 can be seen as an isomorphism

$$C^*(\mathbb{P}(E)) \cong \bigoplus_{i=0}^{r-1} C^*(X, f)[2i](i)$$

in the derived category of $\mathbf{M}^{\exp}(k)$. The inclusion $\mathbb{P}(E) \to \mathbb{P}(E \oplus \mathcal{O}_X)$ induces a morphism

$$\bigoplus_{i=0}^{r} C^*(X, f)[2i](i) \to \bigoplus_{i=0}^{r-1} C^*(X, f)[2i](i)$$

which is indeed just the obvious projection, hence an isomorphism

$$C^*(\mathbb{P}(E \oplus \mathcal{O}_X), \mathbb{P}(E), f \circ \pi) = C^*(X, f)[2r](r)$$

as we wanted, in the case where $Y$ is empty. For the general case, observe that the following diagram the columns are exact triangles, and the second and third horizontal map are isomorphisms in the derived category of $\mathbf{M}^{\exp}(k)$.

$$\begin{array}{ccc} C^*(\mathbb{P}(E\oplus \mathcal{O}_X), \mathbb{P}(E)\cup \mathbb{P}(E\oplus \mathcal{O}_X)|_Y, f\circ\pi) & \longrightarrow & C^*(X,Y,f)[2r](r) \\ \downarrow & & \downarrow \\ C^*(\mathbb{P}(E\oplus \mathcal{O}_X), \mathbb{P}(E), f\circ\pi) & \longrightarrow & C^*(X,f)[2r](r) \\ \downarrow & & \downarrow \\ C^*(\mathbb{P}(E\oplus \mathcal{O}_X)|_Y, \mathbb{P}(E)|_Y, f\circ\pi) & \to & C^*(Y,f|_Y)[2r](r) \end{array}$$

The top horizontal morphism is thus an isomorphism in the derived category as well, and yields the sought isomoprhism of motives by taking homology in degree $n$. ☐

Proposition 4.6.11 (Sphere bundle formula). — *Let $X$ be a smooth variety over $k$, let $f\colon X\to \mathbb{A}^1$ be a regular function and let $\Pi\colon E\to X$ be a vector bundle of rank $r$. Denote by $0_X$ the image of the zero section $X\to E$. There is an isomorphism in $\mathbf{M}^{\exp}(k)$ as follows:*

$$H^*(E\setminus 0_X, f\circ\pi)\cong H^n(X,f)\oplus H^{n-2r+1}(X,f)(r).$$

Proof. ☐

## 4.7. Motives with support, Gysin morphism, and proper pushforward

In this section, we show that the Gysin map in rapid decay cohomology is a morphism of motives. This will enable us to construct a duality pairing in the next section, thus completing the proof that exponential motives form a tannakian category.

4.7.1. — Let $(X,f)$ be a variety with potential, and let $Y\subseteq X$ be a closed subvariety. Let $Z\subseteq X$ be another closed subvariety with open complement $U\subseteq X$. The inclusion $U\subseteq X$ induces a morphism

$$C^\bullet(X,Y,f)\longrightarrow C^\bullet(U,U\cap Y,f_{|U}) \tag{4.7.1.1}$$

in $\mathscr{D}^b(\mathbf{M}^{\exp}(k))$. Concretely, choosing cellular filtrations we can see this as an actual morphism of chain complexes in $\mathbf{M}^{\exp}(k)$. We set

$$C^\bullet_Z(X,Y,f)=\operatorname{cone}(C^\bullet(X,Y,f)\longrightarrow C^\bullet(U,U\cap Y,f_{|U})) \tag{4.7.1.2}$$

Definition 4.7.2. — The *exponential motive of* $(X,Y,f)$ *with support on* $Z$ is the homology of the cone of the morphism (4.7.1.1), namely:

$$H^n_Z(X,Y,f)=H^n(C^\bullet_Z(X,Y,f)). \tag{4.7.2.1}$$

By definition, $H^n_Z(X,Y,f)$ fits into a long exact sequence of motives

$$\cdots\to H^n_Z(X,Y,f)\to H^n(X,Y,f)\to H^n(U,Y\cap U,f|_U)\to H^{n+1}_Z(X,Y,f)\to\cdots \tag{4.7.2.2}$$

THEOREM 4.7.3. — *Let $X$ be a smooth, irreducible variety over $k$, together with a regular function $f$, and let $Y$ be a closed subvariety of $X$. For any smooth closed subvariety $Z \subseteq X$ of pure codimension $c$, with open complement $U$, there is a canonical isomorphism of motives*

$$H^n_Z(X,Y,f) \xrightarrow{\sim} H^{n-2c}(Z, Y\cap Z, f_{|Z})(-c). \tag{4.7.3.1}$$

*Under this isomorphism, the long exact sequence* (4.7.2.2) *becomes a long exact sequence of motives*

$$\cdots \to H^n(X,f) \to H^n(U, f|_U) \to H^{n-2c+1}(Z, f|_Z)(-c) \to H^{n+1}(X,f) \to \cdots$$

*whose underlying long exact sequence of vector spaces is the Gysin sequence* (3.7.2.1).

PROOF. We use deformation to the normal cone as in Chapter 5 of [**Ful98**]. Let $\widetilde{X}$ be the blow-up of $Z\times\{0\}$ in $X\times\mathbb{A}^1$, and equip $\widetilde{X}$ with the potential

$$\widetilde{f}\colon \widetilde{X} \to X\times\mathbb{A}^1 \to X \xrightarrow{f} \mathbb{A}^1$$

obtained by composing the blow-up map, the projection to $X$ and $f$. Let $\pi\colon \widetilde{X}\to\mathbb{A}^1$ denote the composition of the blow-up map $\widetilde{X}\to X\times\mathbb{A}^1$ and the projection to $\mathbb{A}^1$. The fibre of $\pi$ over any non-zero point of $\mathbb{A}^1$ is isomorphic to $X$. The fibre $\pi^{-1}(0)$ has two irreducible components, namely $\mathbb{P}(N_Z\oplus\mathcal{O}_Z)$, the projective completion of the normal bundle $N_Z = T_XZ/T_ZZ$ of $Z$ in $X$, and $\mathrm{Bl}_ZX$, the blow-up of $Z$ in $X$. These two components intersect in $\mathbb{P}(N_Z)$, seen as the infinity section in $\mathbb{P}(N_Z\oplus\mathcal{O}_Z)$ and as the exceptional divisor in $\mathrm{Bl}_ZX$.

Let $\widetilde{Y}$ be the strict transform of $Y\times\mathbb{A}^1$ in $\widetilde{X}$. The intersection of $\widetilde{Y}$ with $\pi^{-1}(0)$ also has two components, namely $\widetilde{Y}\cap\mathbb{P}(N_Z\oplus\mathcal{O}_Z) = \mathbb{P}(N_Z\oplus\mathcal{O}_Z)|_{Y\cap Z}$ and $\widetilde{Y}\cap\mathrm{Bl}_ZX$, the strict transform of $Y$ in $\mathrm{Bl}_ZX$.

The inclusions $\pi^{-1}(0)\to\widetilde{X}$ and $X\cong\pi^{-1}(1)\to\widetilde{X}$ induce morphisms of motives as follows:

$$H^n(\pi^{-1}(0), \mathrm{Bl}_ZX\cup\mathbb{P}(N_Z\oplus\mathcal{O}_Z)|_{Y\cap Z}, \widetilde{f}|_{\pi^{-1}(0)}) \xleftarrow{(*)} H^n(\widetilde{X}, \widetilde{Y}\cup\mathrm{Bl}_ZX, \widetilde{f}) \to H^n(X,Y,f)$$

The map labelled $(*)$ is an isomorphism (of vector spaces, hence of motives) because $[\widetilde{X}, \pi^{-1}(0), \widetilde{f}]$ has trivial cohomology. The reason for this is that the quotient space $\widetilde{X}/\pi^{-1}(0)$ is the same as $(X\times\mathbb{A}^1)/(X\times 0)$, which is a cone, hence contractible. We have a morphism

$$\begin{aligned} &H^n(\pi^{-1}(0), \mathrm{Bl}_ZX\cup\mathbb{P}(N_Z\oplus\mathcal{O}_Z)|_{Y\cap Z}, \widetilde{f}|_{\pi^{-1}(0)}) \\ &\qquad\xrightarrow{\cong} H^n(\mathbb{P}(N_Z\oplus\mathcal{O}_Z), \mathbb{P}(N_Z)\cup\mathbb{P}(N_Z\oplus\mathcal{O}_Z)|_{Y\cap Z}, \widetilde{f}|_{\mathbb{P}(N_Z\oplus\mathcal{O}_Z)}) \end{aligned}$$

induced by inclusion, which is an isomorphism by excision. The right-hand side is isomorphic to $H^{n-2c}(Z, Y\cap Z, f|_Z)(-c)$ by Corollary 4.6.10 to the projective bundle formula. We obtain a morphism of motives

$$H^{n-2c}(Z, Y\cap Z, f)(-c) \to H^n(X,Y,f) \tag{4.7.3.2}$$

whose underlying morphism of vector spaces is the Gysin map.

The rest of the argument is formal. The projective bundle formula can be seen as an isomorphism

$$C^*(\mathbb{P}(E)) \cong \bigoplus_{i=0}^{r-1} C^*(X)[2i](i)$$

in the derived category $D^b(\mathbf{M})$. The morphism of pairs $[\pi^{-1}(0), \mathrm{Bl}_Z X] \to [\widetilde{X}, \mathrm{Bl}_Z X]$ induces therefore a morphism

$$C^*(Z)[2c](c) \to C^*(X) \tag{4.7.3.3}$$

in $D^b(\mathbf{M})$ inducing (4.7.3.2). Its composition with $C^*(X) \to C^*(U)$ is zero, hence a morphism

$$C^*(Z)[2c](c) \to C^*_Z(X)$$

in $D^b(\mathbf{M})$. This morphism is indeed an isomorphism, because the underlying morphism in the derived category of $\mathbb{Q}$-vector spaces is so. The Gysin sequence in the statement of the theorem is, via this isomorphism, the sequence of cohomology with support (4.7.2.2). □

4.7.4. — Let $(X, f)$ be a smooth variety together with a regular function, and let $Z \subseteq X$ be a smooth subvariety of pure codimension $c$.

Proposition 4.7.5. — *Let $(X, f_X)$ and $(Z, f_Z)$ be smooth varieties, together with regular functions, and $\pi\colon Z \to X$ a proper morphism such that $f_Z = f_X \circ \pi$. Set $c = \dim X - \dim Z$. The proper pushforward morphism*

$$\pi_*\colon H^n(Z, f_Z) \longrightarrow H^{n+2c}(X, f_X)(c) \tag{4.7.5.1}$$

*is a morphism of exponential motives.*

Proof. It suffices to treat the case where $\pi$ is a closed immersion. Indeed, since $X$ is quasi-projective, choosing a locally closed embedding $Z \hookrightarrow \mathbb{P}^m$, we can factor the morphism $\pi$ into the composite $Z \xhookrightarrow{\iota} X \times \mathbb{P}^m \xrightarrow{p} X$, where $\iota$ is a closed embedding and $p$ is the projection. If we endow $X \times \mathbb{P}^m$ with the function $f_X \boxplus 0$, then both maps are compatible with the functions. Assume that the pushforward $\iota_*$ is a morphism of motives. Then $\pi_*$ is given by the composition

$$H^n(Z, f_Z) \xrightarrow{\iota_*} H^{n+2c+2m}(X \times \mathbb{P}^m, f_X \boxplus 0)(c+m) \longrightarrow H^{n+2c}(X, f_X)(c),$$

where the second morphism is the projection onto the component

$$H^{n+2c}(X, f_X)(c) \otimes H^{2m}(\mathbb{P}^m)(m) = H^{n+2c}(X, f_X)(c)$$

of the Künneth formula.

□

## 4.8. Duality

Let $X$ be a smooth connected variety of dimension $d$, together with a regular function $f$, and $Y \subseteq X$ a normal crossing divisor. We choose a good compactification $(\overline{X}, \overline{Y}, \overline{f})$ of the triple $(X, Y, f)$ in the sense of Definition 3.5.8. We let $D$ denote the complement of $X$ in $\overline{X}$, $P$ the reduced pole divisor of $\overline{f}$, and we write $D = P + H$. We set

$$X' = \overline{X} \setminus (\overline{Y} \cup P), \qquad Y' = H \setminus (H \cap P)$$

and denote by $f'$ the restriction of $\overline{f}$ to $X'$.

$$H^n_{\mathrm{rd}}(X,Y,f)\otimes H^{2d-n}_{\mathrm{rd}}(X',Y',-f')\longrightarrow\mathbb{Q} \tag{4.8.0.2}$$

Proposition 4.8.1. — *There is a unique morphism of exponential motives*

$$H^n(X,Y,f)\otimes H^{2d-n}(X',Y',-f')\longrightarrow\mathbb{Q}(-d) \tag{4.8.1.1}$$

*whose perverse realisation is the duality pairing* (4.8.0.2).

Proof. We first construct a morphism in the opposite direction. For this, let $\Delta = X\cap X'$ embedded diagonally in $X\times X'$. By construction, $\Delta$ does not intersect $(Y\times X')\cup(X\times Y')$ and the function $f\boxplus(-f')$ is identically zero on $\Delta$. Thus, Theorem 4.7.3 yields an isomorphism of motives

$$\mathbb{Q}(-d)=H^0(\Delta)(-d)\xrightarrow{\sim}H^{2d}_{\Delta}(X\times X',(Y\times X')\cup(X\times Y'),f\boxplus(-f')).$$

Composing with the natural "forget support" map and with the projection to the $H^n(X,Y,f)\otimes H^{2d-n}(X',Y',-f')$ component of the Künneth isomorphism, we obtain a morphism of motives

$$\mathbb{Q}(-d)\longrightarrow H^n(X,Y,f)\otimes H^{2d-n}(X',Y',-f').$$

□

Observe that, when $f$ is constant, we recover the usual duality between cohomology and cohomology with compact support. More generally, this suggests to introduce the following definition:

Definition 4.8.2. — Let $X$ be a smooth variety and $f\colon X\to\mathbb{A}^1$ a regular function. We choose a good relative compactification of $X$ over $\mathbb{A}^1$, *i.e.* a smooth variety $X^{\mathrm{rel}}$ such that $H=X^{\mathrm{rel}}\setminus X$ is a normal crossing divisor and $f$ extends to a proper morphism $f^{\mathrm{rel}}\colon X^{\mathrm{rel}}\to\mathbb{A}^1$. The *motive with compact support* of the pair $(X,f)$ is $H^n(X^{\mathrm{rel}},H,f^{\mathrm{rel}})$.

## 4.9. The motivic Galois group

Let us summarise what we did so far. We first constructed $\mathbf{M}^{\exp}(k)$ as an abelian category. Using the basic lemma, we proved that this category is equivalent to the one obtained from the quiver of cellular objects. This enabled as to define a tensor product. We then show that each objects admits a dual, completing the proof that $\mathbf{M}^{\exp}(k)$ is a tannakian category.

Definition 4.9.1. — The *exponential motivic Galois group* $\mathrm{G}^{\exp}(k)$ is the affine group scheme over $\mathbb{Q}$ such that

$$\mathbf{M}^{\exp}(k)=\mathbf{Rep}(G^{\exp}(k)).$$

Given an exponential motive $M$, the smallest Tannakian subcategory $\langle M\rangle^{\otimes}$ of $\mathbf{M}^{\exp}(k)$ containing $M$ is equivalent to $\mathbf{Rep}(G_M)$ for a linear algebraic group $G_M$. We shall call it the Galois group of $M$

Definition 4.9.2. — Let $Q$ be a one-dimensional exponential motive over $k$. A *$Q$-polarisation* of an exponential motive $M$ is a morphism $M \otimes M \longrightarrow Q$ in $\mathbf{M}^{\exp}(k)$ such that the underlying bilinear map is definite.

Lemma 4.9.3. — *If $M$ admits a polarisation, then $G_M$ is a reductive group.*

CHAPTER 5

# Relation with other theories of motives

## 5.1. Relation with Nori motives

Let $\mathrm{Q}(k)$ denote the full subquiver of $\mathrm{Q}^{\mathrm{exp}}(k)$ consisting of those objets $[X, Y, f, n, i]$ where $f$ is the zero function. The restriction of the Betti representation $\rho$ to $\mathrm{Q}(k)$ is given by ordinary relative cohomology:

$$\rho([X, Y, 0, n, i]) = H^n(X(\mathbb{C}), Y(\mathbb{C}))(i).$$

The cohomological, non-effective variant of Nori's category of mixed motives over $k$ may be defined as the category $\mathbf{M}(k) = \langle Q(k), \rho\rangle$. This is not Nori's original construction, but the one Ayoub sketches in [**Ayo14**, p.6]. The inclusion $\mathrm{Q}(k) \to \mathrm{Q}^{\mathrm{exp}}(k)$ can be seen as a morphism of quiver representations, hence induces a faithful and exact functor

$$\iota\colon \mathbf{M}(k) \to \mathbf{M}^{\mathrm{exp}}(k)$$

which permits us to regard classical Nori motives as exponential motives.

THEOREM 5.1.1. — *The functor $\iota\colon \mathbf{M}(k) \to \mathbf{M}^{\mathrm{exp}}(k)$ is fully faithful and exact.*

PROOF. We only need to prove that $\iota$ is full. For this, it suffices to show that, for each object $M$ of $\mathbf{M}(k)$, the following map is surjective.

$$\mathrm{End}_{\mathbf{M}(k)}(M) \to \mathrm{End}_{\mathbf{M}^{\mathrm{exp}}(k)}(\iota(M))$$

Let $M$ be an object of $\mathbf{M}(k)$ and let $f\colon \iota(M) \to \iota(M)$ be an endomorphism in $\mathbf{M}^{\mathrm{exp}}(k)$. Recall from 4.1 that $M$ consists of the data $(V, Q, \alpha)$, where $V$ is a finite-dimensional $\mathbb{Q}$-vector space, $Q$ is a finite subquiver of $Q(k)$ which we suppose to be non-empty to rule out degenerate cases, and $\alpha\colon \mathrm{End}(\rho|_Q) \to \mathrm{End}(V)$ a morphism of $\mathbb{Q}$-algebras. The exponential motive $\iota(M)$ is given by the same triple $(V, Q, \alpha)$, with $Q$ now regarded as a finite subquiver of $\mathrm{Q}^{\mathrm{exp}}(k)$. The morphism $f$ is a linear map $V \to V$ such that there exists a finite subquiver $P \subseteq \mathrm{Q}^{\mathrm{exp}}(k)$ containing Q and such that $f$ is $\mathrm{End}(\rho|_P)$-linear. We need to find a subquiver $\mathrm{Q}'$ of $\mathrm{Q}(k)$ containing Q, such that $f$ is $\mathrm{End}(\rho|_{\mathrm{Q}'})$-linear.

Let $S \subset \mathbb{A}^1(\mathbb{C})$ be the union of the singularities of all perverse realisations of objets in $P$. As Q is non-empty, this set contains $0 \in \mathbb{C}$. We choose $z \in \mathbb{A}^1(k)$ such that $\mathrm{Re}(z) > \mathrm{Re}(s)$ for all $s \in S$. Consider the functor $\lambda_z : P \to \mathrm{Q}(k)$ given by

$$\lambda_z\colon [X, Y, f, n, i] \longmapsto [X, Y \cup f^{-1}(z), 0, n, i]$$

on objects and by the obvious rules on morphisms. After enlarging Q and $P$ by adding all objects and morphisms in the image of $\lambda_z$, we may see $\lambda_z$ as a functor from $P$ to Q. As we have seen in (3.1.2.2), there are isomorphisms of vector spaces

$$H^n_{\mathrm{rd}}(X,Y,f)(i) \cong H^n(X, Y \cup f^{-1}(z))(i) \tag{5.1.1.1}$$

which are functorial for morphisms in $P$. Together with these isomorphisms, $\lambda_z$ is a morphism of quiver representations. It fits into a commutative diagram

$$\begin{array}{ccccc} Q & \xrightarrow{\subseteq} & P & \xrightarrow{\lambda_z} & Q \\ & {\scriptstyle\rho}\searrow & \downarrow{\scriptstyle\rho} & \swarrow{\scriptstyle\rho} & \\ & & \mathbf{Vec}_{\mathbb{Q}} & & \end{array}$$

where the composition of the horizontal arrows is the identity on $Q$. The left hand triangle commutes, and the right hand triangle commutes up to the natural isomorphisms (5.1.1.1). We obtain morphisms of $\mathbb{Q}$-algebras

$$\mathrm{End}(\rho|_{\mathrm{Q}}) \xrightarrow{\text{via (5.1.1.1)}} \mathrm{End}(\rho|_P) \xrightarrow{\text{res}} \mathrm{End}(\rho|_{\mathrm{Q}})$$

whose composite is the identity. The restriction homomorphism $\mathrm{End}(\rho|_P) \to \mathrm{End}(\rho|_Q)$ is thus surjective, and the induced functor from the caterory of $\mathrm{End}(\rho|_Q)$ modules to the category of $\mathrm{End}(\rho|_P)$ modules is full. In particular, the given $\mathrm{End}(\rho|_P)$-linear morphism $f\colon V \to V$ is also $\mathrm{End}(\rho|_Q)$-linear. □

5.1.2. — From now on, we identify the category of classical Nori motives with its image in the category of exponential motives via the fully faithful functor $\iota$. In the course of the proof of Theorem 5.1.1 we have shown that the morphism of proalgebras

$$\mathrm{End}(\rho) \to \mathrm{End}(\rho|_{Q(k)})$$

given by restriction is surjective and, invoking Zorn's lemma, we even see that it has sections. This tells us more than just fullness of the canonical functor $\iota : \mathbf{M}(k) \to \mathbf{M}^{\exp}(k)$.

Proposition 5.1.3. — *The category of classical motives* $\mathbf{M}(k)$ *is stable under taking subobjects and quotients in* $\mathbf{M}^{\exp}(k)$.

Proof. Let $M$ be an object of $\mathbf{M}(k)$ and let $M'$ be a subobject of $M$ in $\mathbf{M}^{\exp}(k)$. Represent $M$ by a triple $(V, Q, \alpha)$, where $V$ is a finite-dimensional $\mathbb{Q}$-vector space, $Q$ is a finite subquiver of $Q(k)$ and $\alpha\colon \mathrm{End}(\rho|_Q) \to \mathrm{End}(V)$ a morphism of $\mathbb{Q}$-algebras. Then, $M'$ is given by a subspace $V'$ of $V$ which is stable under $\mathrm{End}(\rho|_P)$ for some finite $P \subseteq Q^{\exp}(k)$ containing $Q$. As in the proof of Theorem 5.1.1, we may again enlarge $P$ and $Q$ in such a way that the restriction morphism $\mathrm{End}(\rho|_P) \to \mathrm{End}(\rho|_Q)$ is surjective. But then, $V'$ is stable under $\mathrm{End}(\rho|_Q)$ as we wanted. The same argument works for quotients. □

## 5.2. Conjectural relation with triangulated categories of motives

Let $k$ be a subfield of $\mathbb{C}$. We denote by $\pi\colon \mathbb{A}^1_k \to \operatorname{Spec}(k)$ the structure morphism and by $j\colon \mathbb{A}^1_k \setminus \{0\} \to \mathbb{A}^1_k$ the inclusion.

5.2.1. — Let $\mathrm{DExp}(k)$ be the full subcategory of $\mathrm{DM}_{\mathrm{gm}}(\mathbb{A}^1)$ consisting of those objects $M$ satisfying $\pi_* M = 0$.

CONJECTURE 5.2.2. — *If $k$ is a number field, the canonical functor* $\mathrm{DExp}(k) \to D^b(\mathbf{M}^{\exp}(k))$ *is an equivalence of categories.*

REMARK 5.2.3. — It does not seem reasonable to extend this conjecture beyond the case of number fields. Already for classical motives, the functor $\mathrm{DM}_{\mathrm{gm}}(k) \to D^b(\mathbf{M}(k))$ is *not* an equivalence of categories when the field $k$ has infinite transcendence degree over $\mathbb{Q}$, for otherwise it will induce an equivalence between the corresponding ind-categories. Since the forgetful functor from ind-Nori motives to graded vector spaces is conservative, this would imply that the Betti realisation functor is conservative at the level of ind-Voevodsky motives. This is known to be false if $k$ has infinite transcendence degree [**Ayo17**, Lemma 2.4]. In a nutshell, one picks an infinite sequence $(a_n)_{n\in\mathbb{N}}$ of algebraically independent elements of $k^\times$ and thinks of them as maps $a_n\colon \mathbb{Q}(n)[n] \to \mathbb{Q}(n+1)[n+1]$ via the isomorphisms

$$\mathrm{Hom}_{\mathrm{DM}(k)}(\mathbb{Q}(n)[n], \mathbb{Q}(n+1)[n+1]) \cong \mathrm{Hom}_{\mathrm{DM}(k)}(\mathbb{Q}(0), \mathbb{Q}(1)[1]) \cong k^\times \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Then the ind-motive $F = \{\mathbb{Q}(n)[n]\}_{n\in\mathbb{N}}$ has trivial Betti realisation but is non-zero. Indeed, none of the natural maps $\mathbb{Q}(0) \to \mathbb{Q}(n)[n]$ are zero since they correspond to the symbol $\{a_1, \ldots, a_n\}$ in Milnor K-theory $K_n(k)$ via the identification $\mathrm{Hom}_{\mathrm{DM}(k)}(\mathbb{Q}(0), \mathbb{Q}(n)[n]) \cong K_n^M(k) \otimes \mathbb{Q}$. The natural map $\mathbb{Q}(0) \to F$ is thus non-zero. Returning to our setting, since we now have the exponential at disposal, the situation can only get worse. It is conceivable that a similar construction shows that $k$ needs to have finite *dimension* over $\mathbb{Q}$. We thank Martin Gallauer for pointing this to us.

5.2.4. — The category $\mathrm{DExp}(k)$ contains the category $\mathrm{DM}_{\mathrm{gm}}(\mathbb{A}^1)$ via the functor sending a motive $M$ over $k$ to the motive $j_! j^* \pi^* M[1]$ over $\mathbb{A}^1$. In particular, $\mathrm{DExp}(k)$ contains Tate motives $j_! j^* \pi^* \mathbb{Q}(n)$, which are sent to Tate motives in $D^b(\mathbf{M}^{\exp}(k))$ by the canonical functor. Let us see what conjecture 5.2.2 predicts for extensions.

$$\begin{aligned}
& \mathrm{Ext}^1_{\mathbf{M}^{\exp}(k)}(\mathbb{Q}, \mathbb{Q}(n)) && \\
= \; & \mathrm{Hom}_{\mathrm{DExp}(k)}(j_! j^* \pi^* \mathbb{Q}, j_! j^* \pi^* \mathbb{Q}(n)[1]) && \text{by Conjecture 5.2.2} \\
= \; & \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(\mathbb{G}_m)}(\pi^* \mathbb{Q}, \pi^* \mathbb{Q}(n)[1]) && j_! \text{ fully faithful, renaming } \pi \circ j \text{ as } \pi \\
= \; & \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(k)}(\pi_\# \pi^* \mathbb{Q}, \mathbb{Q}(n)[1]) && \pi_\# \text{ left adjoint to } \pi^* \\
= \; & \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(k)}(\mathbb{Q} \oplus \mathbb{Q}(1)[1], \mathbb{Q}(n)[1]) && \text{because } \pi_\# \pi^* \mathbb{Q}(0) = M(\mathbb{G}_m) \\
= \; & \begin{cases} \mathrm{Ext}^1_k(\mathbb{Q}, \mathbb{Q}(n)) & n \neq 1 \\ \mathrm{Ext}^1_k(\mathbb{Q}, \mathbb{Q}(n)) \oplus \mathbb{Q} & n = 1 \end{cases} && \text{by Conjecture 5.2.2}
\end{aligned}$$

Conjecture 5.2.2 predicts thus, that extensions of $\mathbb{Q}$ by $\mathbb{Q}(n)$ in the category of exponential motives all come from extensions of ordinary motives, except in the case $n = 1$, where we should find essentially one additional nonsplit extension

$$0 \to \mathbb{Q}(1) \to M(\gamma) \to \mathbb{Q} \to 0$$

which explains the additional summand $\mathbb{Q}$ in the last line of the computation. We will produce this extension in section 12.8 and call it *Euler-Mascheroni motive*, because (spoiler alert) among its periods is the Euler-Mascheroni constant. A similar computation, noting the fact that $\mathrm{Ext}^q_k(\mathbb{Q}, \mathbb{Q}(n))$ is zero for $q \neq 0, 1$ shows

$$\mathrm{Ext}^2_{\mathbf{M}^{\exp}(k)}(\mathbb{Q}, \mathbb{Q}(n)) = \mathrm{Ext}^1_k(\mathbb{Q}, \mathbb{Q}(n-1))$$

and $\mathrm{Ext}^q_{\mathbf{M}^{\exp}(k)}(\mathbb{Q}, \mathbb{Q}(n)) = 0$ for $q \neq 0, 1, 2$. As it turns out (??), this isomorphism can be described explicitly as follows: Given an extension $0 \to \mathbb{Q}(n-1) \to M(\nu) \to \mathbb{Q} \to 0$, we twist it by $\mathbb{Q}(1)$ and take the Yoneda cup-product with the Euler-Mascheroni motive $M(\gamma)$. We get a four term exact sequence in $\mathbf{M}^{\exp}(k)$

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathbb{Q}(n) & \to & M(\nu)(1) & & \longrightarrow & & M(\gamma) & \longrightarrow \mathbb{Q} \longrightarrow & 1 \\ & & & & & \searrow & & \nearrow & & & \\ & & & & 0 \longrightarrow & & \mathbb{Q}(1) & & \longrightarrow 0 & & \end{array}$$

representing a class in $\mathrm{Ext}^2(\mathbb{Q}, \mathbb{Q}(n))$.

## 5.3. The Grothendieck ring of varieties with potential

Definition 5.3.1. — The *Grothendieck group of varieties with exponential* is the abelian group $K_0(\mathrm{Var}_k^{\exp})$ defined by the following generators and relations:

- generators are pairs $(X, f)$ consisting of a $k$-variety and a regular function $f\colon X \to \mathbb{A}^1_k$,
- relations are of the following three types:
  (a) $(X, f) = (Y, f \circ h)$ for each isomorphism $h\colon Y \to X$;
  (b) $(X, f) = (Y, f_{|Y}) + (U, f_{|Y})$ for each closed subvariety $Y \subseteq X$ with complement $U$;
  (c) $(X \times \mathbb{A}^1, \mathrm{pr}_{\mathbb{A}^1}) = 0$.

Theorem 5.3.2. — *There is a unique ring morphism*

$$\chi\colon K_0(\mathrm{Var}_k^{\exp}) \longrightarrow K_0(\mathbf{M}^{\exp}(k))$$

*such that, for each pair $(X, f)$, one has*

$$\chi((X, f)) = \sum_{n=0}^{2\dim X} (-1)^n H^n_c(X, f). \tag{5.3.2.1}$$

PROOF. If such a morphism exists, then it is unique since we prescribe it on a set of generators. In order to show its existence, we need to check that (5.3.2.1) is compatible with the relations (a), (b) and (c). For the relations (a) this is clear, and for (c) notice that

$$H_c^n(X \times \mathbb{A}^1, \mathrm{pr}_{\mathbb{A}^1}) = 0$$

for all $n$. It remains to prove in the situation of relation (b) the equality

$$\sum_{n=0}^{2\dim X} (-1)^n H_c^n(X, f) = \sum_{n=0}^{2\dim Y} (-1)^n H_c^n(Y, f|_Y) + \sum_{n=0}^{2\dim U} (-1)^n H_c^n(U, f|_U)$$

in $K_0(\mathbf{M}^{\exp}(k))$. Suppose first that $X$ and $Y \subseteq X$ are smooth. We have then an exact sequence

$$0 \to H_c^0(U, f) \to H_c^0(X, f) \to H_c^0(Z, f) \to H_c^1(U, f) \to \cdots$$

in $\mathbf{M}^{\exp}(k)$ which gives the desired relation. $\square$

CHAPTER 6

# The perverse realisation

## 6.1. Construction and compatibility with tensor products

In this section, we construct the perverse realisation functor $R_{\mathrm{perv}}\colon \mathbf{M}^{\exp}(k) \to \mathbf{Perv}_0$ using Nori's universal property, and show that this functor is compatible with tannakian structures. This means, among other things, that there is a natural isomorphism

$$R_{\mathrm{perv}}(M_1 \otimes M_2) \cong R_{\mathrm{perv}}(M_1) \otimes R_{\mathrm{perv}}(M_2)$$

in $\mathbf{Perv}_0$ for all objects $M_1$ and $M_2$ of $\mathbf{M}^{\exp}(k)$, and that this isomorphism is moreover compatible with the unit, the commutativity and the associativity constraints. The verifications are mostly straightforward and not particularly inspiring, and we will not repeat them for other realisations to come later.

## 6.2. The subquotient question in the abstract setting

Let $\rho\colon Q \to \mathbf{Vec}_{\mathbb{Q}}$ be a quiver representation, and write $\mathbf{M} = \langle Q, \rho\rangle$ for its linear hull. Let us call an object $M$ of $\mathbf{M}$ *elementary* if it is of the form $M = \widetilde{\rho}(q)$ for some object $q$ of $Q$, where $\widetilde{\rho}\colon Q \to \mathbf{M}$ is the canonical lift. We know that every object of $\mathbf{M}$ is isomorphic to a subquotient of a finite sum of elementary objects. In general it is not true that every object of $\mathbf{M}$ can be written as a quotient of a sum of elementary objects. We want to give a criterion on the quiver representation $\rho\colon Q \to \mathbf{Vec}_{\mathbb{Q}}$ for this to happen. The dual question about whether every object of $\mathbf{M}$ can be written as a subobject of a sum of elementary objects can be treated in a similar way, so we leave it aside.

We are interested in this admittedly technical question for the following reason. Assume we want to construct some right exact functor $F\colon \mathbf{M} \to \mathbf{A}$ with values in a $\mathbb{Q}$-linear abelian category $\mathbf{A}$, and suppose that we already constructed $F$ on the class of elemenary objects $\widetilde{\rho}(q)$ for $q \in Q$. Nori's universal property will not help us to extend $F$ to the whole category $\mathbf{M}$. If however every object $M$ of $\mathbf{M}$ can be written as a quotient of a sum of elementary objects, then there exists a presentation

$$E_2 \to E_1 \to M \to 0$$

of $M$, where $E_1$ and $E_2$ are sums of elementary objects. This permits us to define the value of $F$ on $M$ as $F(M) = \mathrm{coker}(F(E_2) \to F(E_1))$. Similarly, if we are given two right exact functors $F\colon \mathbf{M} \to \mathbf{A}$ and $G\colon \mathbf{M} \to \mathbf{A}$, and a functor morphism $\alpha\colon F \to G$ which is an isomorphism on elementary objects, then the fact that every object $M$ of $\mathbf{M}$ can be written as a quotient of a sum of elementary objects implies that $\alpha$ is a functor isomorphism.

PROPOSITION 6.2.1. — *Let $Q$ be a category and let $\rho\colon Q \to \mathbf{Vec}_{\mathbb{Q}}$ be a functor viewed as a quiver representation. Granted the following hypotheses, every object $M$ of $\langle Q, \rho\rangle$ is isomorphic to a quotient of a finite sum of elementary objects:*

- *(H1) For every morphism $f\colon q \to q'$ in $Q$, there exists a morphism $f_1\colon q_1 \to q$ such that the kernel of $\rho(f)$ is equal to the image of $\rho(f_1)$.*
- *(H2) For every two morphisms $f_1\colon q_1 \to q$ and $f_3\colon q_2 \to q$, there exists a morphism $f_3\colon q_3 \to q$ such that the image of $\rho(f_3)$ is the intersection of the images of $\rho(f_1)$ and $\rho(f_2)$.*
- *(H3) For every two object $q_1$ and $q_2$ of $Q$, there exists an object $q$, morphisms $\iota_i\colon q_i \to q$ and $\pi_i\colon q \to q_i$, and an isomorphism $\rho(q) \cong \rho(q_1) \oplus \rho(q_2)$ for which $\rho(\iota_i)$ and $\rho(\pi_i)$ become inclusions and projections.*

LEMMA 6.2.2. — *Let $f\colon q \to q$ be an endomorphism in $Q$ and let $\alpha\colon V \to V$ be an endomorphism of a vector space. Set $M = \widetilde{\rho}(q)$. The kernel of the morphism*

$$\delta\colon V \otimes M \to V \otimes M$$

*sending $v \otimes m$ to $v \otimes f(m) - \alpha(v) \otimes m$ is isomorphic to a sum of copies of $M$ and* $\ker f$.

PROOF. Choosing appropriate bases of $V_2$ and $V_1$, it is enough to treat the following three cases:

(1) $V_2 = 0$ and $V_1 = \mathbb{Q}$
(2) $V_2 = \mathbb{Q}$ and $V_1 = 0$
(3) $V_2 = \mathbb{Q}$ and $V_1 = \mathbb{Q}$ and $\alpha$ is the identity

In the first case we have $\ker \delta = M_2$, and in the second case we have $\ker \delta = \ker f$. In the last case, the exact sequence

$$0 \to M_1 \xrightarrow{\iota} M_1 \oplus M_2 \xrightarrow{\delta} M_2 \to 0$$

with $\iota(m_1) = (m_1, f(m_1))$ and $\delta(m_1, m_2) = f(m_1) - m_2$ shows that $\ker \delta$ is $M_1$. □

PROOF OF PROPOSITION 6.2.1. Pick an object $M$ of $\mathbf{M}$, and recall that $M$ is in fact a triple $(M, Q_0, \alpha)$ where $M$ is a finite-dimensional vector space, $Q_0 \subseteq Q$ is a finite subquiver, and $\alpha : \mathrm{End}(\rho|_{Q_0}) \to \mathrm{End}(M)$ is a $\mathbb{Q}$-algebra homomorphism. Set $E = \mathrm{End}(\rho|_{Q_0})$. The map $\alpha$ endows the vector space $M$ with the structure of an $E$-module. This way, every finite-dimensional $E$-module, in particular $E$ itself, is an object of $\mathbf{M}$. There is a surjective morphism of $E$-modules

$$E^n \to M$$

hence if $E$ can be written as a quotient of a finite sum of elementary objects, then the same is true for $M$. Thus, we can and will suppose without loss of generality that $M = E = (E, Q_0, \mathrm{id})$. By definition, $E$ is the equaliser of a pair or morphisms:

$$0 \longrightarrow E \longrightarrow \prod_{q \in Q_0} \mathrm{End}(\rho(q)) \rightrightarrows \prod_{p \to q} \mathrm{Hom}(\rho(p), \rho(q)). \tag{6.2.2.1}$$

The diagram (6.2.2.1) can be interpreted as a diagram of $E$-modules, hence as a diagram in $\mathbf{M}$. We can write it as

$$0 \longrightarrow E \longrightarrow \prod_{q \in Q_0} \rho(q)^\vee \otimes \widetilde{\rho}(q) \rightrightarrows \prod_{p \to q} \rho(p)^\vee \otimes \widetilde{\rho}(q), \tag{6.2.2.2}$$

where $\rho(q)^\vee$ is the vector space dual to $\rho(q)$ and $\widetilde{\rho}(q)$ the elementary object of $\mathbf{M}$ defined by $q \in Q$. By (H3) and induction, there exists an object $q_0 \in Q$ and morphisms $\iota_q \colon q \to q_0$ and $\pi_q \colon q_0 \to q$ for each $q \in Q_0$ such that $\rho(q_0)$ is the product of the spaces $\rho(q)$, and $\rho(\iota)$ and $\rho(\pi)$ are inclusions and projections. Replacing $Q_0$ by the finite quiver with only object $q_0$ and composite morphisms $\iota_q \circ f \circ \pi_p$ does not change $E$, so we may suppose $Q_0$ consists of a single object and endomorphisms of it. Set $V_0 = \rho(q_0)^\vee$ and $M_0 = \widetilde{\rho}(q_0)$. For each endomorphism $f$ of $q_0$, set $E(f) = \ker \delta_f$ for $\delta_f \colon V_0 \otimes M_0 \to V_0 \otimes M_0$ given by

$$\delta_f(v \otimes m) = \rho(f)^\vee(v) \otimes m - v \otimes \rho(m)$$

□

## 6.3. The subquotient question for motives

In this section, all quivers are understood to be subquivers of $\mathrm{Q}^{\exp}(k)$, equipped with the restriction of the representation $\rho$. In the previous section, we worked out a general criterion (Proposition 6.2.1) for every object in the linear hull of a quiver representation to be a quotient of a sum of elementary objects. In this section, we apply this criterion to the quiver representation

$$\rho \colon \mathrm{Q}^{\exp}(k) \to \mathbf{Vec}_{\mathbb{Q}}$$

whose linear hull is the category of exponential motives. The application is not straightforward, since the hypotheses of 6.2.1 are not met by the representation $\rho$. As far as the hypotheses (H1) and (H2) are concerned things are not looking too dark. More troublesome seems the requirement that $Q$ be a category and $\rho$ a functor, and hypothesis (H3) – how to make sense of a disjoint union

$$[X_1, Y_1, f_1, n_1, i_1] \sqcup [X_2, Y_2, f_2, n_2, i_2]$$

when $n_1$ and $n_2$ or $i_1$ and $i_2$ are distinct? Of course if we only had to look at objects $[X_1, Y_1, f_1, n_0, i_0]$ for some fixed integers $n_0$ and $i_0$, and thus only at morphisms of type (a), the future would look brighter.

Proposition 6.3.1 (bright future). — *Let $M$ be an exponential motive. There exist integers $n_0$ and $i_0 \in \mathbb{Z}$ and a finite quiver $Q_0 \subseteq \mathrm{Q}^{\exp}(k)$ which contains only objects of the form $[X, Y, f, n_0, i_0]$ and morphisms of type (a), such that $M$ is a quotient of a power of the motive* $\mathrm{End}(\rho|_{Q_0})$.

Once Proposition 6.3.1 is established, we can prove the following theorem and main result of this section, by verifying that the hypotheses (H1), (H2) and (H3) hold, up to some inconsequential modifications, for the full subquiver of $\mathrm{Q}^{\exp}(k)$ consisting of objects $[X_1, Y_1, f_1, n_0, i_0]$ for fixed integers $n_0$ and $i_0$.

Theorem 6.3.2. — *Let $M$ be an exponential motive.*

*(1) There exists a finite collection of motives of the form $H^n(X, Y, f)(i)$ and a surjective morphism from the sum of these motives onto $M$.*
*(2) There exist a finite collection of motives of the form $H^n(X, Y, f)(i)$ and an injective morphism from $M$ into the sum of these motives.*

We have restricted our attention in the previous section to the problem of writing objects as quotients of elementary objects, and will continue to do so in this section. The second statement of the theorem could probably be proven along the same lines. In the case of motives, we can deduce (2) from (1) by duality, and the fact that the dual of an elementary motive $H^n(X, Y, f)(i)$ is again elementary.

Definition 6.3.3. — Let $Q$ and $Q_0$ be finite subquivers of $\mathrm{Q}^{\exp}(k)$. We say that $Q$ and $Q_0$ are *equivalent* $Q$ if there exists a finite subquiver $Q^+$ of $\mathrm{Q}^{\exp}(k)$ and an isomorphism

$$\alpha : \mathrm{End}(\rho|_{Q_0}) \xrightarrow{\cong} \mathrm{End}(\rho|_Q)$$

of $\mathrm{End}(\rho|_{Q^+})$-algebras.

6.3.4. — Let $\rho|_Q : Q \to \mathbf{Vec}_{\mathbb{Q}}$ be a quiver representation, say the standard representation on some subquiver $Q$ of $\mathrm{Q}^{\exp}(k)$, and let $Q_1$ be a full subquiver of $Q$. Suppose that in some way we dislike the objects in $Q_1$, and want to replace them with some other objects, without changing the endomorphism algebra $\mathrm{End}(\rho|_Q)$. In other words, we wish to find an a quiver $Q_0$ which is equivalent and contains only likeable objects. That may be possible in theory, as follows. We enlarge $Q$ to a quiver $Q^+$ in three easy steps.

Step 1: Start with setting $Q^+ = Q$ and $Q_1^+ = Q_1$. Then, find for each object $q_1$ of $Q_1$ a finite, connected quiver $L(q_1)$ containing $q_1$ and also containing a non-empty connected subquiver $L^{\heartsuit}(q_1)$ consisting of more likable objects, such that the diagram of vector spaces $\rho(L(q_1))$ is a commutative diagram of isomorphisms. For an object $q_1'$ in $L(q_1)$, denote by $\lambda(q_1')$ the isomorphism $\rho(q_1) \to \rho(q_1')$ appearing in $\rho(L(q_1))$. We add to $Q^+$ these quivers $L(q_1)$. We understand here that we have made sure that the only object common to $L(q_1)$ and $Q$ is $q_1$, and that for different objects $p_1$ and $q_1$ in $Q_1$, the quivers $L(p_1)$ and $L(q_1)$ are disjunct. Let $Q_1^+ \subseteq Q^+$ be the full subquiver consisting of $Q_1$ and those objects in $L(q_1)$ which are not likable.

Step 2: Next, for every morphism $f\colon p_1 \to q_1$ in $Q_1$, find and add to $Q^+$ a morphism $f' : p_1' \to q_1'$, where $p_1'$ and $q_1'$ are objects in $L^\heartsuit(p_1)$ and $L^\heartsuit(q_1)$, such that the diagram

$$\begin{CD} \rho(p_1) @>{\rho(f)}>> \rho(q_1) \\ @V{\lambda(p_1')}VV @VV{\lambda(q_1')}V \\ \rho(p_1') @>{\rho(f')}>> \rho(q_1') \end{CD}$$

commutes.

Step 3: Finally, for every morphism $f\colon q \to q_1$ or $f\colon q_1 \to q$ between an object of $Q_1$ and an object of $Q$ not in $Q_1$, find and add to $Q^+$ morphisms $f' : q \to q_1'$ or $f' : q_1' \to q$, where $q_1'$ is some object in $L^\heartsuit(q_1)$ depending on the morphism $f$ at hand, such that the corresponding diagram

$$\begin{array}{ccc} \rho(q) & \xrightarrow{\rho(f)} & \rho(q_1) \\ & \searrow^{\rho(f')} & \downarrow{\scriptstyle \lambda(q_1')} \\ & & \rho(q_1') \end{array} \qquad\qquad \begin{array}{ccc} \rho(q_1) & \xrightarrow{\rho(f)} & \rho(q) \\ {\scriptstyle \lambda(q_1')}\downarrow & \nearrow_{\rho(f')} & \\ \rho(q_1') & & \end{array}$$

commutes.

Let us refer to this situation by saying that what has been added to $Q^+$ is a *clone* of $Q_1$. Denote now by $Q_0 \subseteq Q^+$ the full subquiver obtained from $Q^+$ by deleting all objects of $Q_1^+$ and all morphisms to and from objects in $Q_1^+$. It is straightforward to check that the restriction morphisms

$$\mathrm{End}(\rho|_Q) \leftarrow \mathrm{End}(\rho|_{Q^+}) \to \mathrm{End}(\rho|_{Q_0})$$

are isomorphisms of algebras. In particular, the quivers $Q$ and $Q_0$ are equivalent.

LEMMA 6.3.5. — *They are!*

PROOF. Element of $E := \mathrm{End}(\rho|_Q)$ are tuples $(e_q)_{q\in Q}$ indexed by objects of $Q$, with $e_q \in \mathrm{End}(\rho(q))$, satisfying

$$e_q \circ \rho(f) = \rho(f) \circ e_p \tag{6.3.5.1}$$

for each morphism $f\colon p \to q$ in $Q$. Similarly, elements of $E_0 := \mathrm{End}(\rho|_{Q_0})$ are tuples $(e_{q_0})_{q_0\in Q_0}$. In order to prove that the restriction map $E \to E_0$ is injective, consider an element $e = (e_q)_{q\in Q}$ of $E$ such that $e_{q_0} = 0$ for all $q_0 \in Q_0$, fix an object $q_1 \in Q_1$, and let us show that $e_{q_1}$ is zero. The hypothesis of the lemma applied to the identity morphism of $q_1$ shows that there exists a unique morphism $h\colon q_1 \to q_0$ in $Q$, and this morphism is such that $\rho(h)$ is an isomorphism. The diagram of vector spaces and linear maps

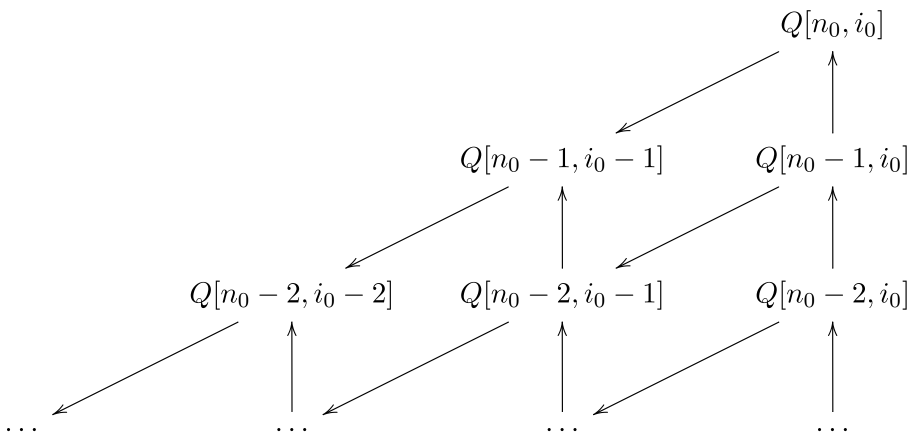
$$\begin{CD} \rho(q_1) @>{\rho(h)}>{\cong}> \rho(q_0) \\ @V{e_{q_1}}VV @VV{e_{q_0}}V \\ \rho(q_1) @>{\rho(h)}>{\cong}> \rho(q_0) \end{CD} \tag{6.3.5.2}$$

commutes, and since $e_{q_0} = 0$ we have indeed $e_{q_1} = 0$. This settles injectivity. In order to prove that the restriction map $E \to E_0$ is also surjective, fix an element $(e_{q_0})_{q_0 \in Q_0}$ of $E_0$. We construct a tuple $(e_q)_{q \in Q}$ by considering as before for each $q_1 \in Q_1$ the unique morphism $h\colon q_1 \to q_0$ whose target is an object in $Q_0$, and take for $e_{q_1}$ the unique endomorphism of $\rho(q_1)$ for which that the square (6.3.5.2) commutes. It remains to pick a morphism $f$ in $Q$ and check that the relation (6.3.5.1) holds. If $f$ is a morphism in $Q_0$, that relation holds because $(e_{q_0})_{q_0 \in Q_0}$ belongs to $E_0$, and if $f$ is a morphism from . □

Proof of Proposition 6.3.1. Given an object $[X, Y, f, n, i]$ of $\mathrm{Q}^{\exp}(k)$, let us refer to the integers $n$ and $i$ as *degree* and *twist* respectively. Finally, given integers $n$ and $i$ and a quiver $Q$, let us denote by $Q[n, i]$ the full subquiver of $Q$ consisting of objects with degree $n$ and twist $i$. Notice that $Q[n, i]$ only contains morphisms of type (a).

Let $Q \subseteq \mathrm{Q}^{\exp}(k)$ be a finite quiver containing objects with different twists and degrees. We will show that there exists a finite quiver $Q_0$ which is equivalent to $Q$ and contains fewer different twists $Q$. We then continue this process until there is only one twist left. We then continue the process further, at each step reducing the number of different degrees and not adding any new twists. This will eventually lead to a quiver which is equivalent to $Q$ and has only one twist and one degree, which proves the proposition.

Since $Q$ is finite, only finitely many of the quivers $Q[n, i]$ are non-empty. Choose $(n_0, i_0)$ large enough, such that whenever $Q[n, i]$ non-empty, then $(n_0, i_0) = (n + d + t, i + t)$ for nonnegative integers $d$ and $t$. The quiver $Q$ can then drawn schematically as a finite diagram of the shape




where the vertical arrows symbolise morphisms of type (b) and the diagonal arrows symbolise morphisms of type (c), and where all nodes are finite quivers with internal morphisms only of type (a). Let us now proceed with reducing the number of twists in $Q$ as outlined above, by proving the following claim.

**Claim:** *Let $i_1$ be the smallest integer such that $Q[n, i_1]$ is non-empty for some $n \leqslant n_0$, and suppose $i_1 < i_0$. There exists a finite quiver $Q_0$ which is equivalent to $Q$ and such that if $Q_0[n, i]$ is non-empty, then $i_1 < i \leqslant i_0$ and $n \leqslant n_0$.*

To prove this claim, let us denote by $Q[i_1]$ the full subquiver of $Q$ of objects with twist $i_1$. We will construct $Q^+$ by adding to $Q$ a clone $Q'$ of $Q[i_1]$ as informally outlined in 6.3.4. As for the first step, let $Q^+$ be the quiver, subject to further enlargement, obtained from $Q$ by adding for every $q = [X, Y, f, n, i_1]$ of $Q[i_1]$ the quiver $L(q)$ consisting of the two objects $q$ and

$$Tq = [X \times \mathbb{G}_m, (Y \times \mathbb{G}_m) \cup (X \times \{1\}), f \boxplus 0, n+1, i_1+1]$$

and the morphism $\kappa_q : Tq \to q$ of type (c). The induced linear map $\rho(\kappa_q) : \rho(Tq) \to \rho(q)$ is an isomorphism, and notice that objects in $Q^+$ have twists between $i_1$ $i_0$ and degrees at most $n_0$. The construction of $Tq \to q$ is functorial in the evident way for morphisms $f\colon q \to q'$ in $Q[i_1]$ of type (a) and (b), so that the diagram of vector spaces

$$\begin{array}{ccc} \rho(q) & \xrightarrow{\cong} & \rho(Tq) \\ {\scriptstyle \rho(f)}\big\downarrow & & \big\downarrow{\scriptstyle \rho(Tf)} \\ \rho(q') & \xrightarrow{\cong} & \rho(Tq') \end{array}$$

commutes. As for the second step in the cloning process, add the morphisms $Tf$ to $Q^+$. For the final step, whenever there is a morphism in $Q$ between an object $q_1$ of $Q[i_1]$ and an object $q$ not in $Q[i_1]$, this morphism must be a morphism $q \to q_1$ of type (c). Thus $q$ is a copy of $Tq_1$, and we add the canonical isomorphism $q = Tq_1$ to $Q^+$. Now we can define $Q_0 \subseteq Q^+$ to be the full subquiver obtained by deleting objects in $Q[i_1]$. As we have checked in Lemma 6.3.5, the algebra morphisms

$$\mathrm{End}(\rho|_Q) \xleftarrow{\cong} \mathrm{End}(\rho|_{Q'}) \xrightarrow{\cong} \mathrm{End}(\rho|_{Q_0})$$

are isomorphisms, and by construction $Q_0$ contains no objects with twist $\leqslant i_1$. This proves the claim, and we can continue the proof of the proposition under the assumption that $Q$ contains only objects with twist $i_0$. We proceed with reducing the number of degrees in $Q$ as outlined above, by proving the following claim.

**Claim:** *Let $n_1$ be the smallest integer such that $Q[n_1, i_0]$ is non-empty, and suppose $n_1 < n_0$. There exists finite quiver $Q_0$ which is equivalent to $Q$ and such that if $Q_0[n, i]$ is non-empty, then $i = i_0$ and $n_1 < n \leqslant n_0$.*

To prove this claim, we will construct $Q^+$ again by adding to $Q$ a clone of the full subquiver $Q[n_1] := Q[n_1, i_0]$. Given an object $q = [X, Y, f, n_1, i_0]$ of $Q[n_1]$ let us denote by $Hq$ and $Dq$ the objects

$$\begin{aligned} Hq &= [(X \times \{0,1\}) \cup (Y \times \mathbb{A}^1), (X \times \{1\}) \cup (Y \times \mathbb{A}^1), f \boxplus 0, n, i_0] \\ Dq &= [X \times \mathbb{A}^1, (X \times \{0,1\}) \cup (Y \times \mathbb{A}^1), f \boxplus 0, n+1, i_0] \end{aligned}$$

and let us write $\iota_q : Hq \to q$ for the morphism of type (a), given by the inclusion of $X = X \times \{0\}$ into $X \times \{0,1\} \cup (Y \times \mathbb{A}^1)$ and $\delta_q : Hq \to Dq$ for the unique morphism of type (b). The morphisms $\rho(\iota_q) : \rho(Hq) \to \rho(q)$ and $\rho(\delta_q) : \rho(Hq) \to \rho(Dq)$ are isomorphisms. For every morphism

$$[X, Y, f, n_1, i_0] \to [W, X, f_W, n_1+1, i_0]$$

of type (b) in $Q$ with source $q$ and target $q'$, let us consider the object

$$E(q,q') = [W \times \mathbb{A}^1, (W \times \{1\}) \cup (X \times \{0\}) \cup (Y \times \mathbb{A}^1), f \boxplus 0, n_1 + 1, i_0]$$

and the morphism $\varepsilon_{q,q'} : E(q,q') \to D(q)$ of type (a) given by the inclusion of $X \times \mathbb{A}^1$ into $W \times \mathbb{A}^1$. The linear map $\rho(\varepsilon_{q,q'})$ is an isomorphism. Let $Q^+$ be the quiver, subject to further enlargement, obtained by adding to $Q$ the objects and morphisms

$$L(q) \quad = \quad \left[\begin{array}{ccccccc} E(q,q') & \searrow^{\varepsilon_{q,q'}} & & & & & \\ E(q,q'') & \xrightarrow{\varepsilon_{q,q''}} & Dq & \xleftarrow{\delta_q} & Hq & \xrightarrow{\iota_q} & q \\ \cdots & \nearrow & & & & & \end{array}\right]$$

for $q \in Q[n_1]$, with one tail on the right hand side for each morphism of type (b) with source $q$. The construction of the objects $Hq$ and $Dq$ and morphisms $\delta_q$ and $\iota_q$ is in the obvious way functorial for morphisms $f\colon q \to q'$ in $Q[n_1]$, which are all of type (a), and the diagram of vector spaces

$$\begin{array}{ccccc} \rho(Dq) & \xleftarrow{\cong} & \rho(Hq) & \xrightarrow{\cong} & \rho(q) \\ {\scriptstyle \rho(Df)}\downarrow & & {\scriptstyle \rho(Hf)}\downarrow & & {\scriptstyle \rho(f)}\downarrow \\ \rho(Dq') & \xleftarrow{\cong} & \rho(Hq') & \xrightarrow{\cong} & \rho(q') \end{array}$$

commutes. As for the second step in the cloning process, add for every morphism $f$ in $Q[n_1]$ the morphism $Df$ to $Q^+$. For the final step, whenever there is a morphism in $Q$ between an object $q$ of $Q[n_1]$ and an object $q'$ not in $Q[n_1]$, this morphism must be a morphism $q \to q'$ of type (b). Add then to $Q^+$ the morphism $E(q,q') \to q'$

$$[W \times \mathbb{A}^1, (W \times \{1\}) \cup (X \times \{0\}) \cup (Y \times \mathbb{A}^1), f \boxplus 0, n_1 + 1, i_0] \to [W, X, f_W, n_1 + 1, i_0]$$

given by the map $W \to W \times \mathbb{A}^1$ sending $w$ to $(w, 0)$.

□

Lemma 6.3.6. — *Let $Q$ be a finite subquiver of $\mathrm{Q}^{\exp}(k)$ and suppose that there exist integers $n_0$ and $i_0$ such that all objects in $Q$ are of degree $n_0$ and twist $i_0$. There exists a squiver $Q_0$ which is equivalent to $Q$ and consists of only*

*(1) one object $q_0$ and endomorphisms, or, alternatively*

*(2) two objects $q_0$ and $q_1$, and besides identities only morphisms of type (a) from $q_0$ to $q_1$, given by morphisms of varieties which are closed immersions. Moreover, one of the morphisms $q_0 \to q_1$ is given by a homotopy equivalence.*

Proof. For notational convenience, we index objects of $Q$ by a finite set, $\mathrm{Obj}(Q) = (q_\alpha)_{\alpha \in A}$, and write $q_\alpha = [X_\alpha, Y_\alpha, f_\alpha, n_0, i_0]$ for every $\alpha \in A$. By adding to each of the varieties $X_\alpha$ and $Y_\alpha$ an isolated point $x_0 = y_0 = \operatorname{Spec} k$ and declare $f(x_0) = 0$, we obtain an equivalent quiver, where now each variety is equipped with a base point. This done, define $q_0 = [X_0, Y_0, f_0, n_0, i_0]$ to be the object obtained from the pair of varieties

$$X_0 = (\operatorname{Spec} k) \sqcup \coprod_{\alpha \in A} X_\alpha \qquad \text{and} \qquad Y_0 = (\operatorname{Spec} k) \sqcup \coprod_{\alpha \in A} Y_\alpha$$

equiped with the function $f_0$, which coincides with $f_\alpha$ on the component $X_\alpha$ and takes the value 0 on the additional point. Let us construct a quiver $Q^+$ obtained by adding to $Q$ the object $q_0$ and the following morphisms:

(1) For each $\alpha \in A$, the morphism $q_0 \to q_\alpha$ given by the inclusion $\iota_\alpha : X_\alpha \to X$.
(2) For each $\alpha \in A$, the morphism $q_\alpha \to q_0$ given by the morphism $\pi_\alpha : X \to X_\alpha$ which is the identity on $X_\alpha$ and sends all components $X_\beta$ with $\beta \neq \alpha$ to the base point of $X_\alpha$.
(3) For each morphism $h\colon q_\alpha \to q_\beta$ in $Q$ given by a morphism of algebraic varieties $h\colon X_\beta \to X_\alpha$, the endomorphism $q_0 \to q_0$ given by the composite $\iota_\beta \circ h \circ \pi_\alpha$.

The vector space $\rho(q_0)$ is the direct sum

$$\rho(q_0) = \bigoplus_{\alpha \in A} \rho(q_\alpha)\,,$$

the morphisms $\rho(\iota_\alpha) : \rho(q_0) \to \rho(q_\alpha)$ are projections and the morphisms $\rho(\pi_\alpha) : \rho(q_\alpha) \to \rho(q_0)$ are the inclusions. The endomorphisms of the object $q_0$ induce, besides the identity, the linear endomorphisms

$$\rho(q_0) \xrightarrow{\text{proj.}} \rho(q_\alpha) \xrightarrow{\rho(h)} \rho(q_\beta) \xrightarrow{\text{incl.}} \rho(q_0)$$

for every morphism $h$ of $Q$, in particular projectors are obtained from identity morphisms $\mathrm{id}_{q_\alpha}$. It is clear that to give an endomorphism of $\rho(q_0)$ which commutes with all these linear endomorphisms is the same as to give an endomorphism of the representation $\rho|_Q : Q \to \mathbf{Vec}_{\mathbb{Q}}$, or more precisely, that the algebra morphisms

$$\operatorname{End}(\rho|_Q) \leftarrow \operatorname{End}(\rho|_{Q^+}) \to \operatorname{End}(\rho|_{Q_0})$$

are isomorphisms. This settles the first variant of the proposition. For the second variant, we may suppose that $Q$ already contains only one object $q_0$ with endomoprhisms. Alternatively, if we introduce a duplicate $q_1$ of $q_0$, and replace each endomorphism of $q_0$ with a morphism $q_0 \to q_1$, we may suppose that $Q$ contains only two objects $q_0$ and $q_1$, and that besides the identities each morphism in $Q$ has source $q_0$ and target $q_1$. Such a morphism

$$h\colon [X_0, Y_0, f_0, n_0, i_0] \to [X_1, Y_1, f_1, n_0, i_0]$$

is given given by a morphism of algebraic varieties $h\colon X_1 \to X_0$ which is compatible with the given subvarieties and regular functions. Using Jouanolou's trick, we may suppose that $X_1$ is affine, hence admits a closed embedding $e : X_1 \to \mathbb{A}^N$. We obtain the sought quiver by replacing $q_0$ by

$$[X_0 \times \mathbb{A}^N, Y_0 \times \mathbb{A}^N, f_0 \boxplus 0, n_0, i_0]$$

and each the morphism $h\colon X_1 \to X_0$ by the closed immersion $(h, e) : X_1 \to X_0 \times \mathbb{A}^N$. □

Proposition 6.3.7. — *Let $h_1 : [X_1, Y_1, f_1, n, i] \to [X_2, Y_2, f_2, n, i]$ be a morphism of type (a) in $\mathrm{Q}^{\exp}(k)$. Suppose that $X_1$ and $X_2$ are affine, and that $h_1$ is given by a closed immersion $X_2 \to X_1$.*

*There exists a morphism $h_0 : [X_0, Y_0, f_0, n, i] \to [X_1, Y_1, f_1, n, i]$ of type (a) such that the sequence of vector spaces*

$$H^n(X_0, Y_0, f_0)(i) \xrightarrow{\rho(h_0)} H^n(X_1, Y_1, f_1)(i) \xrightarrow{\rho(h_1)} H^n(X_2, Y_2, f_2)(i)$$

*is exact.*

Proof. The twist $(i)$ is irrelevant to the question at hand, so we drop it from the notation. Since we suppose that $h_1$ is a closed immersion, we may as well pretend that the morphism $h_2 : X_2 \to X_1$ is the inclusion of a closed subvariety and also think of $Y_2$ as a closed subvariety of $X_1$. In the special case where $Y_2$ is the intersection of $X_2$ and $Y_1$ the proposition is immediate. Indeed, in that case there is an exact sequence

$$\cdots \to H^n(X_1, X_2 \cup Y_1, f_1) \to H^n(X_1, Y_1, f_1) \to H^n(X_2, Y_2, f_2) \to H^{n+1}(X_1, X_2 \cup Y_1, f_1) \to \cdots$$

where morphisms between cohomology groups of the same degree are induced by inclusions. We reduce the general case to this special case by choosing a function $\alpha_1 : X_1 \to \mathbb{A}^N$ such that $\alpha_1^{-1}(0) = Y_2$. Write $\alpha_2$ for the restriction of $\alpha_1$ to $X_2$. We obtain the following diagram of closed immersions

$$\begin{array}{ccc} \mathrm{Graph}(\alpha_2) & \longrightarrow & X_1 \times \mathbb{A}^N \\ \uparrow & & \uparrow \\ Y_2 \times \{0\} & \longrightarrow & Y_1 \times \{0\} \end{array}$$

which is indeed a cartesian square: The intersection of $\mathrm{Graph}(\alpha_2)$ and $Y_1 \times \{0\}$ is $Y_2 \times \{0\}$. By the previously solved case, there exists (and we could give an explicit example of) a tuple $[X_0, Y_0, f_0]$ and a morphism $h_0' : X_1 \times \mathbb{A}^N \to X_0$ compatible with subvarieties and potentials, such that in the following diagram the upper row is exact.

$$\begin{array}{ccccc} H^n(X_0, Y_0, f_0) & \xrightarrow{\rho(h_0')} & H^n(X_1{\times}\mathbb{A}^N, Y_1{\times}\{0\}, f_1{\boxplus}0) & \longrightarrow & H^n(\mathrm{Graph}(\alpha_2), Y_2{\times}\{0\}, f_2{\boxplus}0) \\ & \searrow^{\rho(h_0)} & \cong\uparrow & & \cong\uparrow \\ & & H^n(X_1, Y_1, f_1) & \xrightarrow{\rho(h_1)} & H^n(X_2, Y_2, f_2) \end{array}$$

The vertical morphisms are induced by the projections $X_1 \times \mathbb{A}^N \to X_1$ and $\mathrm{Graph}(\alpha_2) \to X_2$, so the square in the diagram commutes. These projections are homotopy equivalences, so the vertical maps are indeed isomorphisms. A homotopy inverse to the projection $X_1 \times \mathbb{A}^N \to X_1$ is the inclusion $X_1 = X_1 \times \{0\} \to X_1 \times \mathbb{A}^N$. Hence if we choose for $h_0$ the composite of $h_0'$ with this inclusion, the whole diagram commutes, and the image of $\rho(h_0)$ equals the kernel of $\rho(h_1)$. □

Proposition 6.3.8. — *Let $h, h_0 : q_1 \to q_2$ be morphisms of type (a) in $\mathrm{Q}^{\mathrm{exp}}(k)$ which are given by a closed immersions, and suppose $h_0$ is given by a homotopy equivalence. Let $\alpha : V \to V$ be a linear endomorphism of a rational vector space. There exists a morphism $g : p \to q_1$ of type (a) given by a closed immersion, and a linear map $\beta : W \to V_1$ such that the sequence*

$$W \otimes \rho(p) \xrightarrow{\beta\otimes\rho(g)} V \otimes \rho(q_1) \xrightarrow{1\otimes\rho(h)-\alpha\otimes\rho(h_0)} V \otimes \rho(q_2)$$

*is exact.*

PROOF. By Proposition 6.3.7, there exists a □

## 6.4. The theorem of the fixed part

The inclusion $\mathbf{M}(k) \to \mathbf{M}^{\exp}(k)$ of the category of ordinary motives into the category of exponential motives has a right adjoint

$$\Gamma\colon \mathbf{M}^{\exp}(k) \to \mathbf{M}(k)$$

associating with an exponential motive $M$ the largest ordinary submotive $M_0 \subseteq M$. The functor $\Gamma$ is left exact. Similarly, the inclusion of the category of vector spaces into $\mathbf{Perv}_0$ has a right adjoint

$$\Gamma\colon \mathbf{Perv}_0 \to \mathbf{Vec}$$

associating with an object $V$ the largest constant subobject, that is, invariants under the tannakian fundamental group. The perverse realisation $M_0$ is contained in the invariants of the perverse realisation of $M$, hence a natural, injective map $\tau_M\colon R_{\mathrm{B}}\Gamma M \to \Gamma R_{\mathrm{perv}} M$. We can regard this map as a morphism of functors $\tau\colon R_{\mathrm{B}} \circ \Gamma \to \Gamma \circ R_{\mathrm{perv}}$. The theorem of the fixed part states that $\tau$ is an isomorphism, so the square diagram of categories and functors

$$\begin{array}{ccc} \mathbf{M}^{\exp}(k) & \xrightarrow{\ \Gamma\ } & \mathbf{M}(k) \\ {\scriptstyle R_{\mathrm{perv}}}\downarrow & \overset{\tau}{\swarrow} & \downarrow{\scriptstyle R_{\mathrm{B}}} \\ \mathbf{Perv}_0 & \xrightarrow{\ \Gamma\ } & \mathbf{Vec} \end{array} \tag{6.4.0.1}$$

commutes up to an isomorphism of functors $\tau$.

THEOREM 6.4.1. — *Let $M$ be an exponential motive with perverse realisation $V$, and denote by $V_0 \subseteq V$ the largest trivial subobject of $V$. There exists an ordinary motive $M_0$ and an injection $M_0 \to M$ such that the image of the perverse realisation of $M_0$ in $V$ is equal to $V_0$.*

6.4.2 (Caveat). — We will show in a first step that the statement of Theorem 6.4.1 holds for exponential motives of the form $M = H^n(X, Y, f)(i)$. The theorem in its full generality does not follow from this particular case. We know that every exponential motive is isomorphic to a subquotient of a sum of motives of that particular shape. It is also easy to see that if the statement of Theorem 6.4.1 holds for an exponential motive $M$, then it holds for every subobject of $M$, and if the statement holds for $M_1$ and $M_2$, then it holds for $M_2 \oplus M_2$. Quotients are the problem—there is no easy relation between the largest ordinary submotive $M_0$ of $M$ and the largest ordinary submotive of a quotient of $M$. Duals won't help. This problem begs the following question:

6.4.3 (Question). — Is it true or false that every exponential motive is isomorphic to a submotive of a sum of motives of the form $M = H^n(X, Y, f)(i)$?

6.4.4. — Here is an overview on the proof of Theorem 6.4.1. In a first step, we construct a functor or quiver representation

$$h\colon \mathrm{Q}_c^{\exp}(k) \to \mathscr{D}^b(\mathbf{M})$$

which sends a cellular tuple $[X, Y, f, n, i]$ to the object $C^*(X_0, Y_0)[n](i)$, with $X_0 = X \times (\mathbb{A}^1 \setminus \{0\})$ and $Y_0 = \{(y, z) \mid y \in Y \text{ or } f(y) = z\}$. Here, as in section 4.3 (see 4.3.7), the object $C^*(X_0, Y_0)$ in $\mathscr{D}^b(\mathbf{M})$ is obtained from a cellular filtrations of the pair $(X_0, Y_0)$. We are able to construct $h$ very explicitely, first as a functor with values in the category of two term chain complexes in $\mathbf{M}$, as we will describe an explicit cellular filtration on $(X_0, Y_0)$. By construction, the functor $h$ is exact, in the sense that it sends exact sequences to exact triangles.

In a second step, we show that there is a canonical isomorphism of functors $R_{\mathrm{B}} \circ h \cong R\Gamma \circ R_{\mathrm{perv}}$. For motives of the form $M = H^n(X, Y, f)(i)$ we already have a canonical isomorphism

$$R_{\mathrm{B}}(h(M)) \cong R\Gamma R_{\mathrm{perv}}(M)$$

from the Leray-Serre spectral sequence. For general motives, a dévissage argument can be used. Here, it is crucial that we work with derived functors, and not just the left exact $\Gamma$.

In a last step, we show that the functors $h$ and $R\Gamma\colon \mathbf{M}^{\exp} \to \mathscr{D}^b(\mathbf{M})$ are the same. This follows indeed formally from the previous steps.

6.4.5. — Let us put in place a few useful notations. We denote by $\mathrm{Q}_c^{\exp}$ the quiver of cellular objects $[X, Y, f, n, i]$, and by

$$\rho\colon \mathrm{Q}_c^{\exp} \to \mathbf{Vec}$$

the standard representation.

Proposition 6.4.6. — *Let $C$ be an object of $\mathscr{D}\mathbf{M}^{\exp}$ which is quasi-isomorphic to an object of $\mathscr{D}\mathbf{M}$. There is an isomorphism*

$$R\Gamma C \cong C \oplus C[-1](-1)$$

*in $\mathscr{D}\mathbf{M}$.*

Proof. There exists an extension $0 \to \mathbb{Q} \to M(\gamma) \to \mathbb{Q}(-1) \to 0$ in $\mathbf{M}^{\exp}$, whose perverse realisation $R_{\mathrm{perv}} M(\gamma)$ is is given by the local system on $\mathbb{A}^1 \setminus \{0\}$ with local monodromy operator $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ around 0. This particular extension is the subject of section 12.8. Tensoring $M$ with this sequence and applying the functor $R\Gamma$ yields a long exact sequence starting with

$$0 \to \Gamma M \xrightarrow{\cong} \Gamma(M \otimes M(\gamma)) \xrightarrow{0} \Gamma M(-1) \xrightarrow{\cong} R^1\Gamma M \xrightarrow{0} R^1\Gamma(M \otimes M(\gamma)) \to \cdots.$$

We will show that the morphisms are isomorphisms or zero as indicated, and that $R^pM = 0$ for $p \geqslant 2$. The second zero and the fact that $R^pM = 0$ for $p \geqslant 2$ will show that $R\Gamma M$ is given by the two term complex

$$h(M) := [\Gamma(M \otimes M(\gamma)) \xrightarrow{0} \Gamma M(-1)]$$

in degrees 0 and 1, and the statement of the proposition follows then from the fact that the differential in this complex is zero.

Let $\mathbf{M}_0^{\exp} \subseteq \mathbf{M}^{\exp}$ be the smallest full subcategory which contains the image of $c\colon \mathbf{M} \to \mathbf{M}^{\exp}$ and which is stable under extensions. The functor $c\colon \mathbf{M} \to \mathbf{M}^{\exp}$ factors over $\mathbf{M}_0^{\exp}$ by definition, and since $\mathbf{M}_0^{\exp}$ is stable under kernels, cokernels and extensions, its derived category $\mathscr{D}\mathbf{M}_0^{\exp}$ is a full subcategory of $\mathscr{D}\mathbf{M}^{\exp}$. Consider the functors

$$c\colon \mathscr{D}\mathbf{M} \to \mathscr{D}\mathbf{M}_0^{\exp} \qquad \text{and} \qquad h\colon \mathscr{D}\mathbf{M}_0^{\exp} \to \mathscr{D}\mathbf{M}$$

where $h$

□

6.4.7. — Let $q = [X, Y, f, n, i]$ be an object of $\mathrm{Q}_c^{\exp}(k)$, and fix a real $r \gg 0$ which is larger than the real part of any critical value of the function $f$. We construct a two term complex $h_r(q) \in \mathscr{C}^2(\mathbf{M})$ as follows. Set $X^0 := X \times (\mathbb{A}^1 \setminus 0)$ and define the following subvarieties of $X^0$.

$$\begin{aligned} Y^0 &:= \{(x,t) \mid x \in Y \text{ or } f(x) = t\} \\ X_r^0 &:= X \times \{r\} \\ Y_r^0 &:= X_r^0 \cap Y^0 \end{aligned}$$

There is a long exact sequence in $\mathbf{M}$ obtained from the filtration $\varnothing \subseteq X_r^0 \subseteq X^0$ of $X^0$, part of which is the following.

$$\begin{array}{ccccccc} H^n(X^0, Y^0 \cup X_r^0) & \longleftarrow & \cdots & & & & \\ \downarrow & & & & & & \\ H^n(X^0, Y^0) & \longrightarrow & H^n(X_r^0, Y_r^0) & \longrightarrow & H^{n+1}(X^0, Y^0 \cup X_r^0) & \rightarrow & H^{n+1}(X^0, Y^0) \\ & & & & & & \downarrow \\ & & & & \cdots & \longleftarrow & H^{n+1}(X_r^0, Y_r^0) \end{array}$$

We let

$$h_r(q) = [H^n(X_r^0, Y_r^0)(i) \to H^{n+1}(X^0, Y^0 \cup X_r^0)(i)]$$

be the connecting morphism in this sequence, with the appropriate twist. With the obvious constructions for morphisms of type (a), (b) or (c), we can turn $h_r$ into a quiver representation

$$h_r\colon Q \to \mathscr{C}^2(\mathbf{M})$$

on any finite subquiver $Q$ of $\mathrm{Q}_c^{\exp}(k)$, as long as $r$ is large enough. We will now define a similar functor, associating with each object of $\mathbf{Perv}_0$ a two-term complex of vector spaces. Let $F$ be an object of $\mathbf{Perv}_0$, and pick $r \gg 0$ larger than the real part of any singularity of $F$. Let

$$\begin{aligned} j\colon &\quad \mathbb{A} \setminus \{0\} \to \mathbb{A}^1 \\ \alpha\colon &\quad \{r\} \to \mathbb{A}^1 \setminus \{0\} \\ \beta\colon &\quad \mathbb{A}^1 \setminus \{r, 0\} \to \mathbb{A}^1 \setminus \{0\} \end{aligned}$$

be inclusions. The short exact sequence $0 \to \beta_! \beta^* j^* F \to j^* F \to \alpha_* \alpha^* j^* F \to 0$ of sheaves on $\mathbb{A}^1 \setminus 0$ induces the following six term exact sequence of cohomology groups.

$$\begin{array}{ccccccc} H^0(\mathbb{A}^1 \backslash \{0\}, \beta_! \beta^* j^* F) & \longleftarrow & 0 & & & & \\ \downarrow & & & & & & \\ H^0(\mathbb{A}^1 \backslash \{0\}, F) & \longrightarrow & H^0(\mathbb{A}^1 \backslash \{0\}, \alpha_* \alpha^* j^* F) & \to & H^1(\mathbb{A}^1 \backslash \{0\}, \beta_! \beta^* j^* F) & \longrightarrow & H^1(\mathbb{A}^1 \backslash \{0\}, F) \\ & & & & & & \downarrow \\ & & & & 0 & \longleftarrow & H^1(\mathbb{A}^1 \backslash \{0\}, \alpha_* \alpha^* j^* F) \end{array}$$

We let

$$h_r(F) = [H^0(\mathbb{A}^1 \setminus \{0\}, \alpha_* \alpha^* j^* F) \to H^1(\mathbb{A}^1 \setminus \{0\}, \beta_! \beta^* j^* F)]$$

be the connecting morphism in this sequence. The construction of $h_r(F)$ is functorial.

Lemma 6.4.8. — *Let notations be as in 6.4.7 and set $M = H^n(X, Y, f)(i)$.*

1. *The pair $(X^0_r, Y^0_r)$ is cellular in degree $n$, and the pair $(X^0, Y^0 \cup X^0_r)$ is cellular in degree $n+1$.*
2. *There is a canonical and natural isomorphism $R\Gamma F \cong h(F)$ in the derived category of vector spaces.*
3. *There is a canonical and natural isomorphism $R_B(h_r(q)) \cong h_r(R_{\mathrm{perv}}(M))$ of complexes of vector spaces.*

Proof. □

6.4.9. — Let $q$ and $q'$ be objects of $Q$, and let $\varphi\colon M \to M'$ be a morphism between the corresponding motives. The morphism $\varphi$ induces a morphism $\varphi\colon R_{\mathrm{perv}}(M) \to R_{\mathrm{perv}}(M')$ in $\mathbf{Perv}_0$, whence a morphism

$$R_B(h_r(q)) \cong h_r(R_{\mathrm{perv}}(M)) \to h_r(R_{\mathrm{perv}}(M')) \cong R_B(h_r(q')) \tag{6.4.9.1}$$

of complexes of vector spaces. The isomorphisms in (6.4.9.1) are those from Lemma 6.4.8. Our next Lemma states that the linear map (6.4.9.1) is indeed a morphism of complexes of motives.

Lemma 6.4.10. — *Let notations be as in 6.4.9. There is a unique morphism of complexes of motives $h_r(q) \to h_r(q')$ whose Betti realisation is the morphism* (6.4.9.1).

Proof. □

Theorem 6.4.11. — *Let $q = [X, Y, f, n, i]$ be an object of $\mathrm{Q}_c^{\exp}(k)$, set $M = H^n(X, Y, f)(i)$, and let $r \gg 0$ be a real number, larger than the real part of any critical value of $f$. There is an isomorphism*

$$R\Gamma M \cong h_r(q)$$

*in the derived category $\mathscr{D}^b M(k)$.*

Proof. We prove this using the characterisation of $\Gamma$ as a right adjoint. Firs, recall that since $\Gamma : \mathbf{M}^{\exp}(k) \to \mathbf{M}(k)$ is right adjoint to the inclusion $c : \mathbf{M}(k) \to \mathbf{M}^{\exp}(k)$, the derived functor

$$R\Gamma : \mathscr{D}\mathbf{M}^{\exp}(k) \to \mathscr{D}\mathbf{M}(k)$$

is right adjoint to the inclusion $c : \mathscr{D}\mathbf{M}(k) \to \mathscr{D}\mathbf{M}^{\exp}(k)$. See for example [G. Maltsinosis, Quillen's adjunction theorem revisited, ArXiV]. In order to prove that $R\Gamma M$ is isomorphic to $h_r(q)$, we need to establish an isomorphism of vector spaces

$$\mathrm{Hom}_{\mathscr{D}\mathbf{M}^{\exp}(k)}(cM_0, M) \to \mathrm{Hom}_{\mathscr{D}\mathbf{M}(k)}(M_0, h_r(q))$$

which is natural in $M_0$. We take the unit-counit point of view on adjunctions.

We define a unit morphism $\varepsilon$ and a counit morphism $\eta$

$$\begin{aligned} \varepsilon_M : &\quad ch_r(q) \to M \\ \eta_{M_0} : &\quad M_0 \to h_r c M_0 \end{aligned}$$

□

Corollary 6.4.12. — *Let $q = [X, Y, f, n, i]$ be an object of $\mathrm{Q}_c^{\exp}(k)$ and set $M = H^n(X, Y, f)(i)$. The canonical morphism in the derived category of vector spaces*

$$\tau_M : R_{\mathrm{B}} R\Gamma M \to R\Gamma R_{\mathrm{perv}} M$$

*is an isomorphism. In particular the statement of Theorem 6.4.1 holds for the motive $M$.*

Proof of Theorem 6.4.1. Let us recapitulate what we did so far.

Let $Q \subseteq \mathrm{Q}_c^{\exp}(k)$ be a finite subquiver and set $E = \mathrm{End}(\rho|_Q)$. We need to show that the statement of Theorem 6.4.1 holds for every $E$-module $M$. Let us first consider the $E$-module $E$ itself. There is a four term exact sequence

$$0 \to E \to M_1 \to M_2 \to L \to 0 \tag{6.4.12.1}$$

where we know that $\tau_{M_1}$ and $\tau_{M_2}$ are isomorphisms. Let us consider the second half $0 \to K \to M_2 \to L \to 0$ of this four term sequence, and write out the associated long exact sequences.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R_B\Gamma K & \longrightarrow & R_B\Gamma M_2 & \longrightarrow & R_B\Gamma L & \xrightarrow{\partial} & R_B R^1\Gamma K & \longrightarrow & R_B R^1\Gamma M_2 \\
 & & {\scriptstyle (1)}\big\downarrow & & \| & & \big\downarrow & & {\scriptstyle (2)}\big\downarrow & & \| \\
0 & \longrightarrow & \pi_* j^* R_{\mathrm{perv}} K & \longrightarrow & \pi_* j^* R_{\mathrm{perv}} M_2 & \longrightarrow & \pi_* j^* R_{\mathrm{perv}} L & \xrightarrow{\partial} & R^1\pi_* j^* R_{\mathrm{perv}} K & \longrightarrow & R^1\pi_* j^* R_{\mathrm{perv}} M_2
\end{array}$$

As indicated, we already know that the leftmost three vertical maps are injective, and that those vertical maps associated with $M_2$ are isomorphisms. A diagram chase now reveals that (1) is an isomorphism, and (2) is injective. Let us next consider the first half $0 \to E \to M_1 \to K \to 0$ of (6.4.12.1) and write out long exact sequences. Schematically, the diagram takes the following form, where (1) and (2) are the maps appearing in the previous diagram.

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \xrightarrow{\partial} & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & & \\
 & & \big\downarrow & & \| & & (1)\| & & \big\downarrow & & \| & & (2)\big\downarrow & & \\
0 & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \xrightarrow{\partial} & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & 0
\end{array}$$

A diagram chase shows that the first and fourth vertical map, which are $R_B\Gamma E \to \pi_* j^* R_{\mathrm{perv}} E$ and $R_B R^1\Gamma E \to R^1\pi_* j^* R_{\mathrm{perv}} E$, are isomorphisms. In other words,

$$\tau_E : R_B R\Gamma E \to R\pi_* j^* R_{\mathrm{perv}} E$$

is an isomorphism, and, in particular, the statement of Theorem 6.4.1 holds for the motive $E$. For a general finite-dimensional $E$-module $M$, we can choose a free resolution

$$\cdots \to E^{n_2} \to E^{n_1} \to E^{n_0} \to M \to 0$$

and deduce that $\tau_M : R_B R\Gamma M \to R\pi_* j^* R_{\mathrm{perv}} M$ is an isomorphism as well. □

LEMMA 6.4.13. — *An exponential motive of the form* $M = H^n(X \times \mathbb{A}^1, Y \times \mathbb{A}^1, g)(i)$ *with* $g(x,z) = zf(x)$ *for some function* $f\colon X \to \mathbb{A}^1$ *is an ordinary motive.*

PROOF. We can ignore the twist. According to Lemma 4.2.5, there is a canonical isomorphism of motives

$$M = H^n(X \times \mathbb{A}^1, Y \times \mathbb{A}^1, g) \xrightarrow{\cong} H^{n+1}(X \times \mathbb{A}^2, \{(x,z,t) \mid x \in Y \text{ or } zf(x) = t\}, p)$$

where $p$ is the projection given by $p(x,z,t) = t$. We fabricate an ordinary motive $M_0$ from the fibre of $p$ over 1, setting

$$M_0 = H^n(X \times \mathbb{A}^1, Y \times \mathbb{A}^1 \cup \{(x,z) \mid x \in Y \text{ or } zf(x) = 1\}, 0).$$

Lemma 4.2.5 applied to the motive $M_0$ yields the isomorphism

$$M_0 \cong H^{n+1}(X \times \mathbb{A}^2, \{(x,z,t) \mid x \in Y \text{ or } zf(x) = 1 \text{ or } t = 0\}, p)$$

With this presentation of $M_0$, a morphism $\varphi^* : M \to M_0$ is given by the map

$$\begin{array}{ccc}
X \times \mathbb{A}^2 & \xrightarrow{\varphi} & X \times \mathbb{A}^2 \\
\{(x,z,t) \mid x \in Y \text{ or } zf(x) = 1 \text{ or } t = 0\} & \longrightarrow & \{(x,z,t) \mid x \in Y \text{ or } zf(x) = t\}
\end{array}$$

given by $\varphi(x,z,t) = (x, zt, t)$. We claim that $\varphi^*\colon M \to M_0$ is an isomorphism. This can be checked on perverse realisations. □

PROOF OF THEOREM 6.4.1. Without loss of generality we suppose $i = 0$ and suppress the twist from the notation. We give a geometric construction of a morphism $\varphi\colon M_0 \to M$ with the required properties. We do not care about injectivity, because we can always render $\varphi$ injective by replacing $M_0$ by $\varphi(M_0)$. Set $M_0 := H^n(X \times \mathbb{A}^1, Y \times \mathbb{A}^1, g)$ with $g(x,z) = zf(x)$, and define

$$M_0 = H^n(X \times \mathbb{A}^1, Y \times \mathbb{A}^1, g) \xrightarrow{\varphi} H^n(X, Y, f) = M$$

to be the morphism induced by the inclusion $x \longmapsto (x,1)$ of $X$ into $X \times \mathbb{A}^1$. By Lemma 6.4.13 we already know that $M_0$ is an ordinary motive. Let $\varphi\colon V_0 \to V$ be the perverse realisation of $\varphi\colon M_0 \to M$. We have to prove that the image of $\varphi\colon V_0 \to V$ is the largest trivial subobject of $V$.

□

## 6.5. Applications of Gabber's torus trick

Right at the beginning of the book [**Kat90**], Katz lists several fairly general—yet extremely powerful—results from representation theory which later on become the main tools to determine monodromy groups. It is not surprising that these results are useful to understand the fundamental groups in the tannakian category $(\mathbf{Perv}_0, \Phi)$. As we shall see later (where?), there is a direct link between the monodromy groups of differential equations computed in [**Kat90**] and the tannakian fundamental groups of objects of $\mathbf{Perv}_0$.

6.5.1 (Results from representation theory). —

THEOREM 6.5.2 (Gabber's torus trick, cf. Theorem 1.0 in [**Kat90**]). — *Let $\mathfrak{g} \subseteq \mathfrak{gl}_n$ be a semisimple Lie algebra acting irreducibly on $\mathbb{C}^n$. Let $K$ be a torus and let $\chi_1, \ldots, \chi_n : K \to \mathbb{G}_m$ be characters of $K$ corresponding to a homomorphism $\chi : K \to \mathrm{GL}_n$ to the diagonal of $\mathrm{GL}_n$. Suppose that the conjugation action of $K$ on $\mathfrak{gl}_n$ leaves $\mathfrak{g}$ invariant. Let $\mathfrak{t} \subseteq \mathfrak{gl}_n$ be the subspace of those diagonal matrices whose diagonal entries $t_1, \ldots, t_n$ satisfy*

*(1)* $t_1 + t_2 + \cdots + t_n = 0$
*(2)* $t_i - t_j = t_k - t_m$ *whenever* $\chi_i\chi_j^{-1} = \chi_k\chi_m^{-1}$.

*The $\mathfrak{t}$ is contained in $\mathfrak{g}$.*

THEOREM 6.5.3 (Kostant). — *Let $\mathfrak{g} \subseteq \mathfrak{gl}_n$ be a semisimple Lie algebra acting irreducibly on $\mathbb{C}^n$. If $\mathfrak{g}$ contains the diagonal matrix $\mathrm{diag}(n-1, -1, \ldots, -1)$, then $\mathfrak{g}$ is $\mathfrak{sl}_n$.*

THEOREM 6.5.4. — *Let $F$ be a Lie-simple object of $\mathbf{Perv}_0$ of rank $n$, with set of singularities $S$ of cardinality $n$. Suppose that, for any four not necessarily distinct singularities $s_1$, $s_2$, $s_3$, $s_4$ of $F$, the relation $s_1 + s_2 = s_3 + s_4$ implies $\{s_1, s_2\} = \{s_3, s_4\}$. Then, the Lie algebra of the tannakian fundamental group of $F$ contains $\mathfrak{sl}_n$. It is equal to $\mathfrak{sl}_n$ if and only if the sum of all singularities of $F$ is zero.*

6.5.5 (The generic Galois group). — Here is a statement that we learnt from Will Sawin, Theorem 6.5.6. — *Let $f \in k[x]$ be a polynomial of degree $n$. Assume that the following two conditions hold:*

*(i) The derivative $f'$ has no multiple roots.*
*(ii) Given four roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ of $f'$ in $\mathbb{C}$, not necessarily distinct, the equality of complex numbers $f(\alpha_1) + f(\alpha_2) = f(\alpha_3) + f(\alpha_4)$ implies $\{\alpha_1, \alpha_2\} = \{\alpha_3, \alpha_4\}$.*

*Then the motivic Galois group of $H^1(\mathbb{A}^1, f)$ equals $\mathrm{GL}_{n-1}$.*

Proof. □

CHAPTER 7

# The comparison isomorphism revisited

Let $X$ be a variety defined over a field $k \subseteq \mathbb{C}$, let $Y \subseteq X$ a closed subvariety and let $f$ a regular function on $X$. In this chapter, we introduce the de Rham cohomology $H^n_{\mathrm{dR}}(X, Y, f)$ and define a *period pairing*

$$H_n^{\mathrm{rd}}(X, Y, f) \times H^n_{\mathrm{dR}}(X, Y, f) \longrightarrow \mathbb{C} \tag{7.0.6.1}$$

of which we have already given examples in the introduction using the elementary point of view on rapid decay homology. In the case where $f = 0$, this pairing is the same as the usual period pairing between singular homology and de Rham cohomology. Neither rapid decay homology nor de Rham cohomology changes when we replace $f$ by $f + c$ for some constant $c$. The period pairing will change! The main result of this section is Theorem 7.4.1 which states that the period pairing (7.0.6.1) is non-degenerate, hence induces an isomorphism

$$H^n_{\mathrm{dR}}(X, Y, f) \otimes_k \mathbb{C} \xrightarrow{\cong} H^n_{\mathrm{rd}}(X, Y, f) \otimes_{\mathbb{Q}} \mathbb{C} \tag{7.0.6.2}$$

of complex vector spaces. This result is due to Hien and Roucairol, see [**HR08**, Thm. 2.7]. The overall structure of the proof, of which we give a simplified and self contained version here, is similar to that of the classical proof in the case where $f = 0$. It relies on a Poincaré Lemma which we state as Theorem 7.3.14, and a GAGA argument which permits to compare algebraic with analytic de Rham cohomology.

## 7.1. The algebraic de Rham cohomology of varieties with potential

In this section we introduce the algebraic de Rham cohomology of pairs of varieties with potential. For smooth varieties the definition is straightforward, and was already given in the introduction. As for ordinary de Rham cohomology there are several ways of extending the definition to singular varieties, which all lead to the same result [**HMS17**, Chap. 3]. We adopt here the point of view of hypercoverings which we have introduced already in Section 4.5. We fix a field $k$ of characteristic zero. All varieties and morphisms are tacitly supposed to be defined over $k$, and we write $\mathbb{A}^1$ for the affine line $\operatorname{Spec} k[x]$ over $k$.

7.1.1. — Let $(X, f)$ be a pair consisting of a smooth algebraic variety $X$, and a regular function $f \colon X \to \mathbb{A}^1$. Let $d_f$ be the integrable connection on the trivial vector bundle $\mathcal{O}_X$ defined by

$d_f(1) = -df$ or, equivalently, by

$$\begin{aligned} d_f\colon \mathcal{O}_X &\longrightarrow \Omega^1_X \\ g &\longmapsto d_f(g) = dg - gdf \end{aligned}$$

on local sections $g$ of $\mathcal{O}_X$. Note that $d_f$ depends only on $df$ and, for constant $f$, agrees with the usual differential. The connection canonically extends to a complex

$$\mathcal{O}_X \xrightarrow{d_f} \Omega^1_X \xrightarrow{d_f} \cdots \xrightarrow{d_f} \Omega^d_X,$$

where $d = \dim X$ and $d_f\colon \Omega^p_X \to \Omega^{p+1}_X$ is given by $d\omega - df \wedge \omega$ on local sections $\omega$ of $\Omega^p_X$.

Definition 7.1.2. — The *de Rham cohomology* of the pair $(X, f)$ is the cohomology

$$H^n_{\mathrm{dR}}(X, f) = H^n(X, (\Omega^\bullet_X, d_f)).$$

7.1.3. — If $f$ is constant, we recover from definition 7.1.2 the usual algebraic de Rham cohomology of $X$. There is a spectral sequence

$$E_2^{p,q} = H^p(X, \Omega^q_X) \implies H^{p+q}_{\mathrm{dR}}(X, f)$$

which degenerates if $X$ is affine. Therefore, if $X$ is affine, the de Rham cohomology of $(X, f)$ in degree $n$ is the homology of the complex of global differential forms $(\Omega^\bullet_X(X), d_f)$ in degree $n$. For not necessarily affine $X$, this gives a way to compute $H^n_{\mathrm{dR}}(X, f)$ as follows. Given a covering $(U_i)_{i\in I}$ of $X$ by open affine subvarieties, define $X_p$ for integers $p \geqslant 0$ to be the disjoint union of the $U_{\sigma(0)} \cap \cdots \cap U_{\sigma(n)}$ as $\sigma$ ranges over all maps $\sigma : [n] \to I$. Together with the inclusion maps obtained from nondecrasing maps $[m] \to [n]$, the $X_n$ form a simplicial scheme $X_\bullet$. Denote by $f_n$ the restriction of $f$ to each component of $X_n$. We obtain a double complex of vector spaces

$$\begin{array}{ccccc} \vdots & & \vdots & & \\ \downarrow & & \downarrow & & \\ \mathcal{O}_{X_1}(X_1) & \xrightarrow{d_{f_1}} & \Omega^1_{X_1}(X_1) & \xrightarrow{d_{f_1}} & \cdots \\ \downarrow & & \downarrow & & \\ \mathcal{O}_{X_0}(X_0) & \xrightarrow{d_{f_0}} & \Omega^1_{X_0}(X_0) & \xrightarrow{d_{f_0}} & \cdots \end{array} \tag{7.1.3.1}$$

whose vertical differentials are defined to be alternating sums of face maps $X_n \to X_{n+1}$. The associated total complex represents $R\Gamma(X, (\Omega^\bullet_X, d_f))$ in the derived category of vector spaces, hence in particular computes the de Rham cohomology of $(X, f)$.

7.1.4. — Let $X$ be a possibly singular variety, together with a regular function $f$. Let $X_\bullet \to X$ be a smooth affine hypercovering of $X$ and let $f_n$ be the function induced on each $X_n$. We say that $(X_\bullet, f_\bullet)$ is a smooth affine hypercovering of $(X, f)$. Each face $\delta_i\colon X_n \to X_{n+1}$ induces by functoriality morphisms of coherent sheaves $\Omega^p_{X_n} \to \delta_i^*\Omega^p_{X_{n+1}}$ which commute with $d_f$, hence a complex of sheaves $(\Omega^\bullet_{X_\bullet}, d_{f_\bullet})$ on the simplicial scheme $X_\bullet$. Face maps induce morphisms

$\Omega^p_{X_n}(X_n) \to \Omega^p_{X_n}(X_{n-1})$. By considering the alternating sums of these morphisms, we obtain a double complex which looks just like (7.1.3.1). We denote by $R\Gamma_{\mathrm{dR}}(X_\bullet, f_\bullet)$ the associated total complex. It is a consequence of the next lemma that once we regard $R\Gamma_{\mathrm{dR}}(X_\bullet, f_\bullet)$ as an object of the derived category of $k$-vector spaces, then, up to a unique isomorphism, it only depends on $(X, f)$ and not on the chosen affine hypercovering. In particular, definition 7.1.6 is unambiguous.

LEMMA 7.1.5. — *Let $X$ be a variety together with a regular function $f$. Let $(X_\bullet, f_\bullet) \to (X, f)$ and $(X'_\bullet, f'_\bullet) \to (X, f)$ be smooth affine hypercoverings of $X$, and let $h : (X_\bullet, f_\bullet) \to (X'_\bullet, f'_\bullet)$ be a morphism of hypercoverings. The morphism of complexes of $k$-vector spaces induced by $h$*

$$h^* : R\Gamma_{\mathrm{dR}}(X'_\bullet, f_\bullet) \to R\Gamma_{\mathrm{dR}}(X_\bullet, f_\bullet)$$

*is a quasi-isomorphism, and is independent of $h$ up to homotopy.*

PROOF. Idea: Use SGA 4.2. exp. 1 §7.3.1 to define the notion of homotopy between maps of hypercoverings. Observe that any pair of two maps $h_0$ and $h_1$ defines an elementary homotopy

$$v : (X_\bullet, f_\bullet) \to (X'_\bullet, f'_\bullet) \times \Delta^1$$

Apply the functor $R\Gamma_{\mathrm{dR}}(-)$ in order to get out of this a homotopy between $R\Gamma_{\mathrm{dR}}(h_0)$ and $R\Gamma_{\mathrm{dR}}(h_1)$. □

DEFINITION 7.1.6. — Let $X$ be a variety over $k$, together with a regular function $f$. Let $(X_\bullet, f_\bullet) \to (X, f)$ be a smooth affine hypercovering. We define the de Rham cohomology of $(X, f)$ as

$$H^n_{\mathrm{dR}}(X, f) := H^n(R\Gamma_{\mathrm{dR}}(X_\bullet, f_\bullet)). \tag{7.1.6.1}$$

More generally, given a closed subvariety $Y \subseteq X$, we define relative de Rham cohomology as

$$H^n_{\mathrm{dR}}(X, Y, f) = H^n(\mathrm{cone}(R\Gamma_{\mathrm{dR}}(X_\bullet, f_\bullet) \to R\Gamma_{\mathrm{dR}}(Y_\bullet, (f|_Y)_\bullet))). \tag{7.1.6.2}$$

7.1.7. — Künneth formula for product with $(\mathbb{G}_m, \{1\})$

DEFINITION 7.1.8. — The *de Rham representation* $\rho_{\mathrm{dR}} \colon Q^{\exp}(k) \to \mathbf{Vec}_k$ is given on objects by

$$\rho_{\mathrm{dR}}([X, Y, f, n, i]) = H^n_{\mathrm{dR}}(X, Y, f)(i)$$

and on morphisms as follows:

(a)

(b)

$$\begin{array}{ccccc}
R\Gamma_{\mathrm{dR}}(X, f) & \longrightarrow & R\Gamma_{\mathrm{dR}}(Y, f|_Y) & \longrightarrow & R\Gamma_{\mathrm{dR}}(X, Y, f) \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow \\
R\Gamma_{\mathrm{dR}}(X, f) & \longrightarrow & R\Gamma_{\mathrm{dR}}(Z, f|_Z) & \longrightarrow & R\Gamma_{\mathrm{dR}}(X, Z, f) \\
\downarrow & & \downarrow & & \downarrow \\
\text{acyclic} & \longrightarrow & R\Gamma_{\mathrm{dR}}(Y, Z, f|_Y) & \longrightarrow & \text{cone}
\end{array}$$

Using the nine lemma, we get a triangle

$$R\Gamma_{\mathrm{dR}}(X,Y,f) \longrightarrow R\Gamma_{\mathrm{dR}}(X,Z,f) \longrightarrow R\Gamma_{\mathrm{dR}}(Y,Z,f|_Y)$$

and morphisms of type $(b)$ are sent to the connecting morphism in the corresponding long exact exact sequence of cohomology

$$\cdots \longrightarrow H^{n-1}_{\mathrm{dR}}(Y,Z,f|_Y) \longrightarrow H^n_{\mathrm{dR}}(X,Y,f) \longrightarrow \cdots$$

(c)

7.1.9. — De Rham cohomology is compatible with extension of scalars.

7.1.10. — Let $X$ be a smooth projective complex variety. Every Zariski-open subset of $X$ is open for the analytic topology, and every regular function on a Zariski-open set of $X$ is analytic, hence a continuous map $X^{\mathrm{an}} \to X$ and a morphism of sheaves of rings $s^*\mathcal{O}_X \to \mathcal{O}_X^{\mathrm{an}}$. Serre's GAGA theorems [**Ser56**, Theorems 1,2,3] state that the analytification functor

$$(-)^{\mathrm{an}} : \{\text{Coherent } \mathcal{O}_X\text{-modules}\} \to \{\text{Coherent } \mathcal{O}_X^{\mathrm{an}}\text{-modules}\}$$

sending a coherent $\mathcal{O}_X$-module $F$ to $F^{\mathrm{an}} = s^*F \otimes_{s^*\mathcal{O}_X} \mathcal{O}_X^{\mathrm{an}}$ is an equivalence of categories. A particular aspect of this is that for any coherent sheaf $F$ on $X$ the canonical morphisms

$$H^n(X,F) \to H^n(X^{\mathrm{an}}, s^*F) \to H^n(X^{\mathrm{an}}, F^{\mathrm{an}})$$

obtained from $s$ are isomorphisms, and this continues to hold when in place of a single coherent sheaf $F$ we put a complex of coherent sheaves. The differentials in such a complex need not be $\mathcal{O}_X$-linear. The most important example for this situation is the algebraic de Rham complex $\Omega_X^\bullet$ with its usual exterior differential, and its analytification $\Omega_X^{\mathrm{an},\bullet}$. We obtain the canonical isomorphisms

$$H^n_{\mathrm{dR}}(X) = H^n(X, \Omega_X^\bullet) \to H^n(X^{\mathrm{an}}, \Omega_X^{\mathrm{an},\bullet})$$

comparing algebraic and analytic de Rham cohomology. The GAGA theorems fail catastrophically if $X$ is not projective. We can still obtain an easy comparison isomorphism between algebraic and analytic de Rham cohomology for smooth quasiprojective varieties, at the price of choosing a smooth compactification. Let $D$ be a divisor on the smooth projective complex variety $X$. We can compute the algebraic de Rham cohomology of $X_0 = X \setminus D$ as the cohomology on $X$ of the de Rham complex $\Omega_X^\bullet[*D] = \mathcal{O}_X[*D] \otimes \Omega_X^\bullet$ of rational differential forms with poles of arbitrary order in $D$. This complex is just the direct image of the algebraic de Rham complex of $X_0$. The analytification of $\Omega_X^\bullet[*D]$ is the complex $\Omega_X^{\mathrm{an},\bullet}[*D]$ of meromorphic differential forms on $X$ with poles of arbitrary order in $D$. It is not the direct image of the analytic de Rham complex of $X_0$, but rather of the subcomplex of forms of moderate growth. We obtain isomorphisms

$$H^n_{\mathrm{dR}}(X_0) = H^n(X, \Omega_X^\bullet[*D]) \to H^n(X^{\mathrm{an}}, \Omega_X^{\mathrm{an},\bullet}[*D]) \tag{7.1.10.1}$$

using that sheaf cohomology commutes with colimits, and writing $\Omega_X^p[*D] = \operatorname{colim}_m \Omega_X^p[mD]$ as a colimit of coherent sheaves. Grothendieck's theorem [**Gro66**, Theorem 1'] comparing algebraic and analytic de Rham cohomology of $X_0$ relies then on resolution of singularities in order to reduce to a normal crossings divisor $D$, and an explicit computation by Atiyah and Hodge. When working with a variety $X_0$ equipped with a potential $f : X_0 \to \mathbb{A}^1$, we content ourselves for the moment with the isomorphism (7.1.10.1) obtained from GAGA. The differential operator $d_f$ is well defined on algebraic or analytic differential forms on $X$ with poles in $D = X \setminus X_0$, hence the following proposition is immediate.

Proposition 7.1.11. — *Let $X_0$ be a smooth complex variety, and let $f_0$ be a regular function on $X_0$. Let $X$ be a smooth compactification of $X$. The analytification functor induces natural isomorphisms of complex vector spaces*

$$H^n_{\mathrm{dR}}(X_0, f_0) = H^n(X, (\Omega^\bullet_X[*D], d_f)) \to H^n(X^{\mathrm{an}}, (\Omega^{\mathrm{an},\bullet}_X[*D], d_f))$$

## 7.2. Construction of the comparison isomorphism

In this section, we construct the period pairing (7.0.6.1) and the period map (7.0.6.2). Since algebraic de Rham cohomology is compatible with extension of scalars as explained in 7.1.9, we may work without loss of generality with complex varieties. For smooth affine varieties marked in smooth subvarieties, the construction of the period pairing is elementary, we have already given examples in the introduction. We shall see that there is essentially a unique way of extending the period isomorphism from this elementary case to a functorial period isomorphism for all pairs of varieties with potential.

7.2.1. — For a general smooth but not necessarily affine variety $X$ with potential $f$, we can in general not represent all elements in the de Rham cohomology of $(X, f)$ by global differential forms. For this reason, and because it will help us to verify functoriality properties of the period pairing later, we shall define a morphism at the level of complexes, from the deRham complex $(\Omega^\bullet_X, d_f)$ to a singular chain complex, which induces the sought period pairing. The problem we immediately face while trying to make sense of this is that rapid decay cohomology of $(X, f)$ is not the cohomology of a cochain complex on $X$. This is what the real blow-up is for: AWe can choose a good compactification of $X$ and consider the real blow-up $B$ of the compactification at the divisor at infinity. The rapid decay cohomology of $(X, f)$ is the cohomology of the pair $[B, \partial^+ B]$, which can be computed as the cohomology of a singular cochain complex on $B$. We will thus rather define a morphism of chain complexes on the compactification of $X$, from the direct image of the deRham complex to the direct image of the singular chain complex.

7.2.2. — Our conventions for singular cochain complexes as follows. Let $X$ be a manifold with boundary, possibly with corners. The group $C_p(X)$ of singular $p$-chains on $X$ is the $\mathbb{Q}$-linear vector space generated by piecewise smooth[1] maps $T : \Delta^p \to X$ where $\Delta^p \subseteq \mathbb{R}^{p+1}$ is the standard $p$-simplex, defined as the convex hull of the set of canonical basis vectors $e_0, e_1, \ldots, e_p$. The differential

[1]this means: piecewise smooth in the relative interior of each face

$d : C_p(X) \to C_{p-1}(X)$ is given by linearity and

$$dT = \sum_{i=0}^{p} (-1)^i (T \circ d^i)$$

where $d^i : \Delta^{p-1} \to \Delta^p$ is induced by the linear map given by $d^i(e_j) = e_j$ for $j < i$ and $d^i(e_j) = e_{j+1}$ for $j \geqslant i$. The resulting complex of rational vector spaces $C_\bullet(X)$ is the singular chain complex associated with $X$, and we call

$$C^\bullet(X) = \mathrm{Hom}(C_\bullet(X), \mathbb{Q})$$

the *singular cochain complex* of $X$. It computes by definition the singular cohomology of $X$ with rational coefficients. The complex $C^\bullet(X)$ depends contravariantly functorially on $X$. In particular, the assignment of the complex $C^\bullet(U)$ to any open $U \subseteq X$ defines a complex of presheaves on $X$, whose sheafification we denote by $C^\bullet_X$. Since $X$ is locally contractibl, this complex of sheaves is a flasque resolution of the constant sheaf $\mathbb{Q}_X$ on $X$, hence $C^\bullet_X(X)$ computes the sheaf cohomology of $X$ with coefficients in $\mathbb{Q}_X$. One can show, using barycentric subdivision, that the canonical map $C^\bullet(X) \to C^\bullet_X(X)$ is a quasi-isomorphism, hence the canonical isomorphism between singular and sheaf cohomology. Given the inclusion of a subspace $\alpha : Y \to X$, there is a canonical morphism of sheaves $C^\bullet_X \to \alpha_* C^\bullet_Y$, ultimately induced by sending a simplex $T : \Delta^p \to Y$ to $\alpha \circ T$. We denote by

$$C^\bullet_{(X,Y)} := \mathrm{cone}(C^\bullet_X \to \alpha_* C^\bullet_Y)$$

its cone, and observe that $C^\bullet_{(X,Y)}$ is a flasque resolution of $\mathbb{Q}_{(X,Y)} = \ker(\mathbb{Q}_X \to \alpha_* \mathbb{Q}_Y)$, hence computes the cohomology of the pair of spaces $(X, Y)$.

7.2.3. — Consider a smooth variety $X$ endowed with a potential $f : X \to \mathbb{A}^1$. Choose a good compactification $\overline{X}$ of it. The given variety $X$ is thus equal to $\overline{X} \setminus D$ for a normal crossings divisor $D$ on $\overline{X}$, and the given potential $f$ extends to a function $f : \overline{X} \to \mathbb{P}^1$. Let $\pi : B \to \overline{X}$ be the real blow-up of $X$ at $D$, and let $f_B : B \to \widetilde{\mathbb{P}}^1$ be the extension of $f$ to $B$. For $b \in \partial B$, either $f_B(b) \in \mathbb{C}$ or $f_B(b)$ lines on the circle at infinity, in which case its real part can either be negative, zero or positive. Set

$$B^0 := B \setminus \{b \in \partial B \mid f_B(b) \in \mathbb{C} \text{ or } \mathrm{Re}(f_B(b)) \leqslant 0\}$$

so $\partial B^0$ consists of those $b \in \partial B$ where $f_B$ takes an infinite value with strictly positive real part. The inclusion of pairs

$$(B^0, \partial B^0) \xrightarrow{\subseteq} (B, \partial^+ B)$$

is a homotopy equivalence, hence so is the morphism of complexes $\pi_* C^\bullet_{(B,\partial^+ B)} \to \pi_* C^\bullet_{(B^0,\partial B^0)}$ on $X$. Both of these complexes compute the rapid decay cohomology of $(X, f)$ by Proposition 3.5.2. On $\overline{X}$ we have the analytic deRham complex $(\Omega^\bullet_X[*D], d_f)$ of meromorphic differential forms with poles of arbitrary order contained in $D$. Its cohomology on $X$ is the deRham cohomology of $(X, f)$ by Proposition 7.1.11. A morphism of complexes of sheaves

$$I : (\Omega^\bullet_X[*D], d_f) \longrightarrow \pi_* C^\bullet_{(B^0,\partial B^0)} \otimes \mathbb{C} \tag{7.2.3.1}$$

is specified by the following data: For every open $U \subseteq X$ in the analytic topology, and every meromorphic $p$-form $\omega \in \Omega_X^\bullet[*D](U)$ a linear map

$$I_U(\omega) : C_p(\pi^{-1}(U)) \oplus C_{p-1}(\pi^{-1}(U) \cap \partial B^0) \to \mathbb{C}\,. \tag{7.2.3.2}$$

This data must be compatible with inclusions of open subsets, and with differentials. Given a $p$-simplex $T : \Delta^p \to U$ and a $(p-1)$-simplex $T' : \Delta^{p-1} \to U \cap \partial B^0$, we set

$$I_U(\omega)(T, T') = \int_T e^{-f}\omega = \int_{\Delta^n} e^{-f\circ T}\, T^*\omega \tag{7.2.3.3}$$

This makes sense, since $e^{-f}\omega$ is everywhere defined on $B^0$, and so $T^*(e^{-f}\omega) = e^{-f\circ T}\, T^*\omega$ is a piecewise smooth differential form on $\Delta^n$. Compatibility with inclusions of open subsets is tautological. Compatibility with differentials is essentially a consequence of Stokes's formula.

Lemma 7.2.4 (Twisted Stokes formula). — *With notations as in 7.2.3, the equality*

$$I_U(d_f\omega)(T, T') = I_U(\omega)(dT + T', dT')$$

*holds.*

Proof. This can be verified by a straightforward computation. Here it is:

$$\begin{aligned} I_U(d_f\omega)(T,T') &= \int_T e^{-f}(d_f\omega) \\ &= \int_T d(e^{-f}\omega) && \text{(definition of } d_f) \\ &= \int_{dT} e^{-f}\omega && \text{(Stokes)} \\ &= \int_{dT} e^{-f}\omega + \int_{T'} e^{-f}\omega && (\text{because } e^{-f\circ T'} = 0) \\ &= I_U(\omega)(dT + T', dT') \end{aligned}$$

□

7.2.5. — It follows from Lemma 7.2.4 that the integration map (7.2.3.1) is a well defined morphism of complexes of sheaves of complex vector spaces on $X$. Taking cohomology yields morphisms of complex vector spaces

$$H^n(X, (\Omega_X^\bullet[*D], d_f)) \longrightarrow H^n(X, \pi_* C^\bullet_{(B^0, \partial B^0)} \otimes \mathbb{C}) \tag{7.2.5.1}$$

which we may interpret as follows: On the left hand side stands the cohomology on the analytic space $X$ with coefficients in the analytic deRham complex $(\Omega_X^\bullet[*D], d_f)$, which is the deRham cohomology of the pair $(X, f)$ by Proposition 7.1.11. On the right hand side stands the rapid decay cohomology of $(X, f)$ with complex coefficients, by Proposition 3.5.2. The map (7.2.5.1) corresponds to a map

$$H^n_{\mathrm{dR}}(X, f) \longrightarrow H^n_{\mathrm{rd}}(X, f) \otimes \mathbb{C} \tag{7.2.5.2}$$

which is the sought period map for $(X, f)$.

Definition 7.2.6. —

## 7.3. Poincaré Lemmas

The proof of the comparison theorem 7.4.1, stating that the comparison morphism from de Rham cohomology to rapid decay cohomology constructed in the previous section is an isomoprhism relies on a Poincaré Lemma, stated as Theorem 7.3.14 below. This Poincaré Lemma relates the twisted de Rham complex for differential forms of moderate growth on the real blow-up with a singular chain complex, and in turn relies on a theorem of Marco Hien, [**Hie07**, Theorem A.1], about the growth behaviour of solutions of certain systems of linear partial differential equations. In *loc.cit.*, this theorem A.1 is stated and proven in a slightly more general setup than what we need here, which makes its proof substantially more difficult, but only in the 2-dimensional case. The case of arbitrary dimension is a straightforward generalisation. For the readers convenience, we reformulate and prove the case we need here, which is Theorem 7.3.6.

This section is organised as follows: In the long Paragraph 7.3.1, meant as an introduction to the general case, we treat the one dimensional case. From 7.3.2 on, we prepare for the proof of Theorem 7.3.6. This theorem has a local setup, so we work with functions and differential equations on a complex polydisk. From 7.3.7 we work in a global setup, obtaining finally the Poincaré Lemma 7.3.14.

7.3.1. — As a warm-up, consider a meromorphic function $f$ on the complex unit disk with only a pole at zero, and a holomorphic function $h$ defined on a thin open sector $U$ around the real half line of the unit disk.

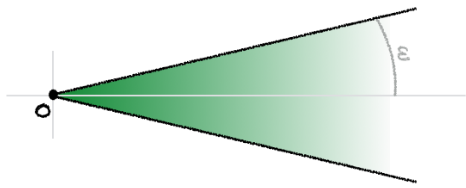



Figure 7.3.1. The open sector $U = \{x \in \mathbb{C} \mid 0 < |x| < 1, \arg(x) < \varepsilon\}$

In what follows we allow ourselves at any moment, if the arguments require it, to replace the sector $U$ by a thinner sector i.e. to choose a smaller $\varepsilon > 0$. The inhomogeneous linear differential equation

$$u' - f'u = h \tag{7.3.1.1}$$

in the unknown function $u$ on $U$ has a one dimensional space of solutions, namely

$$u(x) = e^{f(x)} \int_{1/2}^{x} h(t)e^{-f(t)}dt + Ae^{f(x)} \tag{7.3.1.2}$$

where $A \in \mathbb{C}$ is a constant. The choice of $\frac{1}{2} \in U$ as starting point of the integration path is an arbitrary one. Suppose now that $h$ has moderate growth as $x \to 0$, that means, there exists an integer $N$ such that $|h(x)| = O(|x|^{-N})$ holds for small $x$. Our question is whether the function $u$ has moderate growth as $x \to 0$ for an appropriate choice of the constant $A$. We shall show that this is indeed the case. Writing $f$ as a Laurent series

$$f(x) = ax^{-d} + (\text{terms of degree } > -d)$$

for some integer $d$ and nonzero $a \in \mathbb{C}$, we distinguish four cases. First case: $d \leqslant 0$, so $f$ is holomorphic. In that case, $f$ is bounded around $x = 0$ and $u$ has moderate growth for any choice of $A$. Second case: $d > 0$ and $\mathrm{Re}(a) > 0$. In that case, a special solution of the differential equation is given by the improper integral

$$u(x) = e^{f(x)} \int_0^x h(t)e^{-f(t)}dt \tag{7.3.1.3}$$

which converges, since $e^{-\mathrm{Re}(f(x))}$ decreases exponentially as $x \to 0$. We can use L'Hôpital's rule to examine the behaviour of $u$ near zero: For small $x$ we have

$$u(x) = \frac{\int_0^x h(t)e^{-f(t)}dt}{e^{-f(x)}} \sim \frac{h(x)}{f'(x)}$$

hence $|u(x)| = O(|x|^{-N+d-1}) \leqslant O(|x|^{-N})$ as $x \to 0$. Third case: $d > 0$ and $\mathrm{Re}(a) < 0$. In this case $e^{f(x)}$ converges to 0 as $x \to 0$, hence if $u$ has moderate growth for one choice of $A$, then so it does for any other. We use again L'Hôpital's rule to see that $u(x)$ grows as $\frac{h(x)}{f'(x)}$ as $x$ approaches 0. The difference between this case and the previous one is that now the indeterminacy has the shape $\frac{\infty}{\infty}$ no matter where the integration starts, whereas before it was $\frac{0}{0}$ only because the integration started at $t = 0$. Fourth and last case: $d > 0$ and $\mathrm{Re}(a) = 0$. Assume $a = si$ with real $s > 0$, the case $s < 0$ being similar. A special solution to (7.3.1.1) is again given by the integral formula (7.3.1.3), where the integration path approaches zero from a positive angle $0 < \delta < \varepsilon$. The integral

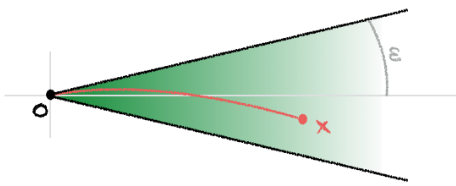



FIGURE 7.3.2. An integration path approaching 0 from a positive angle

converges, since for sufficiently small $\delta$

$$\lim_{r\to 0} r^d \mathrm{Re}(f(re^{\delta i})) = \lim_{r\to 0}(r^d \cdot si \cdot r^{-d} \cdot e^{-d\delta i}) = se^{(\pi/2-d\delta)i}$$

has positive real part. This special solution $u(x)$ has moderate growth of order at most $O(|x|^{-N})$ along any angle above and below the real line as we have seen in the previous cases, hence again $|u(x)| = O(|x|^{-N})$.

Suppose now that instead of solutions of (7.3.1.1) on a sector, we are interested in solutions of defined on the whole pointed disk. Now $f$ and $h$ are both meromorphic functions on the unit disk, with only pole at the origin. As we have seen presently, there exists on each sufficiently small sector around any angle $\alpha$ a solution of moderate growth, and in fact a unique one given by the improper integral (7.3.1.3) if the real part of $f(z)$ tends to $+\infty$ as $z$ approaches zero in the direction of $\alpha$. These local solutions glue together to a global solution if any only if for any two such angles $\alpha$ and $\beta$ the integral

$$\int_{0\cdot\alpha}^{0\cdot\beta} h(t)e^{-f(t)}dt \tag{7.3.1.4}$$

vanishes. The notation means that we integrate along a path in the punctured unit disk starting at 0 in the direction of $\alpha$ and ending at 0 from the direction $\beta$.

We can reformulate our findings in terms of sheaves as follows: Let $\overline{X}$ be the open complex unit disk, set $X = \overline{X} \setminus \{0\}$, and let $\pi : B \to \overline{X}$ be the real blow-up of the origin. Let $f : X \to \mathbb{C}$ be a meromorphic function with only pole at zero, and denote by $f_B : B \to \widetilde{\mathbb{P}}^1$ its extension to the real blow-up. As we have shown in Section 3.5, the rapid decay cohomology of $(X, f)$ is the cohomology of the pair of spaces $[B^\circ, \partial B^\circ]$, where $B^\circ$ is the union (inside $B$) of $X$ and those elements $b$ in $\partial B$ with $f_B(b) \in \partial\widetilde{\mathbb{P}}^1$ with positive real part. The cohomology of the pair $[B^\circ, \partial B^\circ]$ is the cohomology of $B^\circ$ with coefficients in the sheaf $\mathbb{Q}_{[B^\circ,\partial B^\circ]}$. This sheaf admits as a flasque resolution the complex of sheaves $C^\bullet_{[B^\circ,\partial B^\circ]} \otimes \mathbb{C}$ given in degree $p$ by the sheaf of singular cochains on $B^\circ$ with boundary in $\partial B^\circ$. Let $\mathcal{O}^{\mathrm{an}}_{B^\circ}$ denote the sheaf of holomorphic functions on $X = B^\circ \setminus \partial B^\circ$ with moderate growth near $\partial B^\circ$, set $\Omega^{\mathrm{an},1}_{B^\circ} = \mathcal{O}^{\mathrm{an}}_{B^\circ} dx$ and consider the connection $d_f : \mathcal{O}^{\mathrm{an}}_{B^\circ} \to \Omega^{\mathrm{an},1}_{B^\circ}$ sending $u$ to $(u' - f'u)dx$. Integration on chains defines a morphism of complexes of sheaves

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}^{\mathrm{an}}_{B^\circ} & \xrightarrow{d_f} & \Omega^{\mathrm{an},1}_{B^\circ} & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle I_0} & & \downarrow{\scriptstyle I_1} & & \downarrow & & \\
0 & \longrightarrow & C^0_{[B^\circ,\partial B^\circ]}\otimes\mathbb{C} & \xrightarrow{d} & C^1_{[B^\circ,\partial B^\circ]}\otimes\mathbb{C} & \to & C^2_{[B^\circ,\partial B^\circ]}\otimes\mathbb{C} & \longrightarrow & \cdots
\end{array} \tag{7.3.1.5}$$

as follows: A local section $u$ of $\mathcal{O}^{\mathrm{an}}_{B^\circ}$ is sent by $I_0$ to the map which sends a 0-simplex $T : \Delta^0 \to B^\circ$ to the complex number $e^{-f(T(e_0))}u(T(e_0))$, and a local section $hdx$ of $\Omega^{\mathrm{an},1}_{B^\circ}$ is sent $I_0$ to the map which sends a 1-simplex $T : \Delta^1 \to B^\circ$ to the integral

$$I_1(hdx)(T) = \int_T e^{-f} hdx = \int_0^1 e^{-f(T(te_0+(1-t)e_1))} h(T(te_0 + (1-t)e_1))dt$$

with the convention that the standard $n$-simplex is the convex hull in $\mathbb{R}^{n+1}$ of the canonical basis $e_0, e_1, \ldots, e_n$. The kernel of $d_f$ is generated by the function $e^f$ on opens which are disjoint from $\partial B^\circ$ and is zero on opens meeting $\partial B^\circ$, hence the morphism $I_0$ induces an isomorphism of sheaves $\ker(d_f) \to \mathcal{H}^0(C^\bullet_{[B^\circ,\partial B^\circ]}) \otimes \mathbb{C}$. We have shown that also $I_1$ induces an isomorphism $\mathrm{coker}(d_f) \to \mathcal{H}^1(C^\bullet_{[B^\circ,\partial B^\circ]}) \otimes \mathbb{C}$. Therefore (7.3.1.5) is an isomorphism in the derived category of complexes of sheaves

$$I\colon (\Omega^{\mathrm{an},\bullet}_B, d_f) \xrightarrow{\cong} C^\bullet_{[B,\partial^+B]}$$

on $B^\circ$. This is our first local Poincaré Lemma. On $X$, we look at the deRham complex $(\Omega^{\mathrm{an},\bullet}_X[*D], d_f)$ of meromorphic differential forms with a pole of any order at $D = \{0\}$, and the integration morphism

$$I : \Omega^{\mathrm{an},\bullet}_X[*D] \to \pi_* C^\bullet_{[B^\circ,\partial B^\circ]} \otimes \mathbb{C}$$

given by the same formula. Also this morphism is an isomorphism in the derived category. This is our second local Poincaré Lemma. An interesting thing to notice here is that since $C^\bullet_{[B^\circ,\partial B^\circ]}$ is flasque, we could also place the derived direct image $R\pi_* C^\bullet_{[B^\circ,\partial B^\circ]} \otimes \mathbb{C}$ in the above map, and still get an isomorphism. Hence, the canonical morphism

$$\Omega_X^{\mathrm{an},\bullet}[*D] = \pi_*\Omega_{B^\circ}^{\mathrm{an},\bullet} \to R\pi_*\Omega_{B^\circ}^{\mathrm{an},\bullet}$$

is an isomorphism too. This is shown in much greater generality in [**Sab91**, Corollary II, 1.1.8].

7.3.2. — Our goal is to generalise the discussion of 7.3.1 to several variables. We start with the local picture. To this end, we fix the following notation and terminology: pick integers $n > 0$ and $0 \leqslant m \leqslant n$, consider the open unit polydisk $\overline{X} \subseteq \mathbb{C}^n$, the divisor $D$ of $X$ given by $x_1 x_2 \cdots x_m = 0$, and the real blow-up

$$\pi\colon B \to \overline{X}$$

of $\overline{X}$ along the components of $D$. We consider the holomorphic function $f\colon X \to \mathbb{P}^1$ given in projective coordinates by $f(x) = [f_0(x) : f_1(x)]$, with $f_0(x) = x_1^{e_1} x_2^{e_2} \cdots x_m^{e_m}$ for some non-negative integers $e_1, e_2, \ldots, e_m$. The poles $f^{-1}(\infty)$ are contained in $D$, and we denote by $f_B\colon B \to \widetilde{\mathbb{P}}^1$ the continuous extension of $f$ to $B$. Recall that $B$ is the space

$$\{(x_1,\ldots,x_n,w_1,\ldots,w_m) \in \overline{X} \times (S^1)^m \mid x_i w_i^{-1} \in \mathbb{R}_{\geqslant 0} \text{ for } 1 \leqslant i \leqslant m\}$$

and that the boundary $\partial B$ of $B$ is the set $\pi^{-1}(D)$. We identify the interior of $B$ with $X := \overline{X} \setminus D$ via the projection map $\pi$. The real blow up $\widetilde{\mathbb{P}}^1$ of $\mathbb{P}^1$ at $\{\infty\}$ is the complex plane to which a circle at infinity $\partial\widetilde{\mathbb{P}}^1 = S^1$ has been glued. As discussed in 3.5.1, the function $f_B\colon B \to \widetilde{\mathbb{P}}^1$ is given by

$$f_B(x,w) = \begin{cases} \frac{f_1(x)}{f_0(x)} & \in \mathbb{C}, \text{ if } f_0(x) \neq 0 \\[2ex] \frac{f_1(x)}{|f_1(x)|} f_0(w)^{-1} & \in \partial\widetilde{\mathbb{P}}^1, \text{ if } f_0(x) = 0, \text{ hence } (x,w) \in \partial B. \end{cases}$$

In what follows, we are interested in the behaviour of local solutions of differential equations near a point $b \in \partial B$. For our purposes, we may choose a point $b$ in $\pi^{-1}(0)$, thus of the form

$$b = (0,\ldots,0,e^{2\pi i\beta_1},\ldots,e^{2\pi i\beta_m})$$

and consider the open neighbourhoods

$$U = U(\varepsilon) = \big\{(x,w) \bigm| |x_i| < \varepsilon \text{ for } 1 \leqslant i \leqslant n \text{ and } \arg(e^{2\pi i\beta_p} w_p^{-1}) < \varepsilon \text{ for } 1 \leqslant p \leqslant m\big\}$$

of $b$. As a manifold with corners, $U$ is diffeomorphic to $(-1,1)^n \times [0,1)^n$. Concretely, a local chart around $b \in \pi^{-1}(0)$ is given by

$$(-\varepsilon,\varepsilon)^n \times [0,\varepsilon)^n \xrightarrow{\ \simeq\ } U \tag{7.3.2.1}$$

sending $(\alpha_1,\ldots,\alpha_n,r_1,\ldots,r_n)$ to $(x,w) \in B$ with $x_k = r_k e^{2\pi i(\alpha_p+\beta_p)}$ and $w_k = e^{2\pi i(\alpha_p+\beta_p)}$.

Lemma 7.3.3. — *Suppose that $f_B(b) = i\infty \in \partial\mathbb{P}^1$ holds. For sufficiently small $\varepsilon > 0$, the set $\{x \in U \mid \mathrm{Re}(f_B(x)) = 0\}$ divides $U$ into two simply connected components.*

PROOF. First of all, notice that the integers $e_p$ are not all zero - if so, $0 \in \overline{X}$ would not be a pole of $f$, and hence $f_B(b) = f(0)$ would not be an element of the boundary of $\widetilde{\mathbb{P}}^1$. In terms of the coordinates given in (7.3.2.1), the argument of the function $f_B$ is given by

$$\arg(f_B(x, w)) = \arg(f_1(x)) - (e_1(\alpha_1 + \beta_1) + \cdots + e_m(\alpha_m + \beta_m)) \qquad \in \mathbb{R}/2\pi\mathbb{Z}$$

noting that since $f_1(0) \neq 0$, the argument of $f_1$ is a well defined real analytic function in a neighbourhood of 0, taking values in $\mathbb{R}/2\pi\mathbb{Z}$. Set $f_1(x) = f_1(0) \cdot e^{g(x)}$, where $g$ is a holomorphic function satisfying $g(0) = 0$, so that the argument of $f_1$ is given by $\arg(f_1(\alpha, r)) = \arg(f_1(0)) + \operatorname{im}(g(x))$. Writing $(x, w)$ in coordinates $(r, \alpha)$, the Taylor expansion of the function $\operatorname{im}(g(r, \alpha))$ has no linear terms in $\alpha$. We have $\arg(f_B(b)) = \frac{\pi}{2}$ by hypothesis, so we can write $\arg(f_B(x, w))$ as

$$\arg(f_B(x, w)) = \tfrac{\pi}{2} + L(\alpha) + \text{higher order terms}$$

where $L$ is a nonzero linear form, and higher order terms mean terms which contain a factor which is quadratic in $\alpha$ or a factor which is linear in $r$. The cube $(-\varepsilon, \varepsilon)^n \times [0, \varepsilon)^n$ is divided in two halves by the hyperplane $L(\alpha) = 0$, and we deduce from the implicit function theorem that for small enough $\varepsilon > 0$, the cube $(-\varepsilon, \varepsilon)^n \times [0, \varepsilon)^n$ is divided in two halves by the hyperplane $\arg(f_B(x, w)) = \frac{\pi}{2}$. $\square$

DEFINITION 7.3.4. — We say that a holomorphic function $h : U \setminus \partial U \to \mathbb{C}$ has *moderate growth* near $b$ if there exists a neighbourhood $V \subseteq U$ of $b$ and Laurent polynomial $g \in \mathbb{C}[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_m^{-1}]$ such that the inequality $|h(x)| \leqslant |g(x)|$ holds for $x \in V \setminus \partial V$.

7.3.5. — Sums and products of functions with moderate growth again have moderate growth, and, in particular, the function $f$ has moderate growth near $b$. Let us introduce the linear differential operators

$$D_i(u) = \frac{\partial u}{\partial x_i} \qquad \text{and} \qquad Q_i(u) = \frac{\partial u}{\partial x_i} - \frac{\partial f}{\partial x_i} u$$

for $1 \leqslant i < n$. If $h$ has moderate growth near $b$, then so do $D_i(h)$ and $Q_i(h)$.

THEOREM 7.3.6. — *Let $h$ be a holomorphic function on $U \setminus \partial U$ and let $1 \leqslant r \leqslant n$ be an integer. If $h$ satisfies the integrability condition $Q_s(h) = 0$ for all $1 \leqslant s < r$, then the system of partial differential equations*

$$\begin{cases} Q_s(u) = 0 & \text{for } 1 \leqslant s < r \\ Q_r(u) = h \end{cases} \qquad (\Sigma_f(h))$$

*admits a holomorphic solution. If moreover $h$ has moderate growth near $b \in \pi^{-1}(0)$, then there exists a holomorphic solution defined in a neighbourhood of $b$, with moderate growth near $b$.*

PROOF. The difference of any two solutions of $(\Sigma_f(h))$ is a solution of the corresponding homogeneous system $(\Sigma_0(h))$, whose holomorphic solutions form the vector space of functions of the form $Ae^f$, where $A$ is a holomorphic function in the variables $x_{r+1}, \ldots, x_n$. Let us set

$$u(x) = w(x)e^{f(x)} \qquad \text{and} \qquad g = he^{-f}$$

where $w$ stands for a holomorphic function to be determined. We have $Q_s(u) = D_s(w)e^f$, hence must solve the new system

$$\begin{cases} D_s(w) = 0 & \text{for } 1 \leqslant s < r \\ D_r(w) = g \end{cases} \tag{$\Sigma_0(g)$}$$

in the unknown function $w$. The integrability condition on $h$ translates to

$$D_s(g) = \tfrac{\partial h}{\partial x_s} e^{-f} - h \tfrac{\partial f}{\partial x_s} e^{-f} = Q_s(h)e^{-f} = 0$$

for $1 \leqslant s < r$. The differential system $(\Sigma_0(g))$ together with the integrability condition on $g$ is precisely what has to be solved in the proof of the classical Poincaré Lemma. Indeed, the integrability condition on $g$ means that $g$ is constant with respect to the variables $x_1, \ldots, x_{r-1}$, and we can set

$$w(x_1, \ldots, x_n) = \int_{\frac{\varepsilon}{2}}^{x_r} g(x_1, \ldots, x_{r-1}, z, x_{r+1}, \ldots, x_n) dz$$

where the integration path from $\frac{\varepsilon}{2}$ to $x_r$ may be chosen to be a straight line. The general solution $u$ to $(\Sigma_f(h))$ is therefore given by

$$u(x) = e^f \cdot \int_{\frac{\varepsilon}{2}}^{x} he^{-f} dz + Ae^f \tag{7.3.6.1}$$

with the same integration path and some holomorphic function $A$ in the variables $x_{r+1}, \ldots, x_n$. The function $u$ is holomorphic, and all that's left to show is that for some appropriate choice of $A$ the solution $u$ has moderate growth near $b$ if $h$ has so. Let us suppose that this is the case, and choose $\varepsilon < 1$ small enough so that there exists an integer $N \geqslant 0$ for which the inequality

$$|h(x)| \leqslant |x_1 x_2 \cdots x_m|^{-N}$$

holds for $x \in U \setminus \partial U$. We distinguish four possible regimes for $f_B(b) \in \widetilde{\mathbb{P}}^1$, namely $f_B(b)$ can be:

(1) An element of the interior of $\widetilde{\mathbb{P}}^1$. So $f_B(b)$ is a complex number.
(2) An element in the boundary $\partial\widetilde{\mathbb{P}}^1$ with positive real part.
(3) An element in the boundary $\partial\widetilde{\mathbb{P}}^1$ with negative real part.
(4) Either $+i\infty$ or $-i\infty$.

In the first case, the meromorphic function $f = \frac{f_1}{f_0} : \overline{X} \to \mathbb{C}$ is holomorphic and its extension to $B$ is the composite of $f$ with the blow-up map $\pi : B \to \overline{X}$. We may hence assume $f$ is bounded on $U$, say

$$|e^{f(x)}| \leqslant M \qquad \text{and} \qquad |e^{-f(x)}| \leqslant M$$

hold. The function defined by (7.3.6.1) has moderate growth if we choose for $A$ any function of moderate growth, for example a constant. In case (2), the function $e^{-f}$ decays exponentially in a neighbourhood of $b$, and therefore, since $h$ has moderate growth, the integral

$$\int_0^{\frac{\varepsilon}{2}} he^{-f} dz$$

converges. Put differently, we may choose 0 in place of $\frac{\varepsilon}{2}$ as starting point of the integral in (7.3.6.1) even if this new starting point is now on the boundary and not in the interior if $U$. Let us show that the function

$$u(x) = e^{f(x)} \cdot x_r \cdot \int_0^1 h(x_1, \dots, tx_r, \dots, x_n) e^{-f(x_1, \dots, tx_r, \dots, x_n)} dt \tag{7.3.6.2}$$

has moderate growth near $b$. Set $\Phi(x,t) = f(x_1, \dots, x_r, \dots, x_n) - f(x_1, \dots, tx_r, \dots, x_n)$, and notice that for $0 < t < 1$ and all $x \in U$ we have $\mathrm{Re}(\Phi(x,t) < 0$. We may estimate (7.3.6.2) by

$$\begin{aligned} |u(x)| &= \left| x_r \cdot \int_0^1 h(x_1, \dots, tx_r, \dots, x_n) e^{\Phi(x,t)} dt \right| \\ &\leqslant |x_r| \cdot \int_0^1 t \cdot |x_1 x_2 \cdots x_m|^{-N} e^{\mathrm{Re}(\Phi(x,t))} dt \\ &\leqslant |x_1 \cdots x_m|^{-N+1} \end{aligned}$$

which shows that $u$ has moderate growth near $b$ as claimed. Case (3) is similar to case (1): Since $\mathrm{Re}(f)$ tends to $-\infty$ near $b$, the function $e^f$ is bounded near $B$, hence the function defined by (7.3.6.1) has moderate growth near $b$ if we choose for $A$ a constant. The last case which remains to discuss is case (4), in which $f_B(b) = \pm i\infty \in \partial\widetilde{\mathbb{P}}^1$. The boundary of the open $U$ contains the set $\partial^+ U$ given by

$$\partial^+ U := \{(x,w) \in \partial B \mid f_0(x) = 0 \text{ and } \mathrm{Re}(f_B(x,w)) > 0\}$$

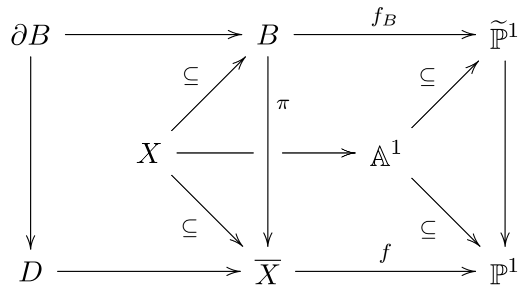
and we will show that $\partial^+ U$ is connected for sufficiently small $\varepsilon$. The rest of the argument will be similar to the case (2).

□

7.3.7. — Our next goal is to reinterpret Theorem 7.3.6 in terms of sheaves of differential forms having *moderate growth* on a smooth complex variety $X$ together with a potential $f$. For the remainder of this section, we work with a fixed smooth and proper complex algebraic variety $\overline{X}$, a normal crossing divisor $D \subseteq \overline{X}$ and a potential $f : \overline{X} \to \mathbb{P}^1$ satisfying $f^{-1}(\infty) \subseteq D$. In other words, writing also $f$ for the restriction of $f$ to $X := \overline{X} \backslash D \to \mathbb{A}^1$, the pair $(\overline{X}, f)$ is a good compactification of $(X, f)$. Let us denote by $\pi : B \to \overline{X}$ the real oriented blow-up of $\overline{X}$ in the components of $D$, and by $\partial B = \pi^{-1}(D)$ the boundary of $B$. The function $f : \overline{X} \to \mathbb{P}^1$ lifts to a function $f_B : B \to \widetilde{\mathbb{P}}^1$ as was shown in 3.5.1. As usual, we set $\partial^+ B := \{b \in \partial B \mid f_B(b) \in \partial\widetilde{\mathbb{P}}^1 \text{ and } \mathrm{Re}(f_B(b)) \geqslant 0\}$ and

$$B^\circ := \{b \in B \mid \pi(b) \notin D, \text{ or } f_B(b) \in \partial\widetilde{\mathbb{P}}^1 \text{ and } \mathrm{Re}(f_B(b)) > 0\}$$

and $\partial B^\circ = B^\circ \cap \partial B$. By Proposition 3.5.2 and its Corollary 3.5.3, the cohomology of the pairs $(B, \partial^+ B)$ and $(B^\circ, \partial B^\circ)$ is canonically isomorphic to the rapid decay cohomology of $(X, f)$. The following diagram summarises the situation.




Since $\overline{X}$ is compact, so is the real blow-up $B$ and its boundary $\partial B$, and the blow-up map $\pi : B \to X$ is proper. Let $U \subseteq B^\circ$ be an open subset, with boundary $\partial U := U \cap \partial B$. Since $U \setminus \partial U$ is an open subset of the complex algebraic variety $B^\circ \setminus \partial B^\circ = X$, it makes sense to speak about algebraic, rational, holomorphic or meromorphic functions on $U \setminus \partial U$. Informally, a function on an open of $B$ has moderate growth if it grows with at most polynomial speed near the boundary. It need not be defined on the boundary but can have a pole there. (Compare with §II, Definition, 2.6 in [**Del70**], or Section 9.2 in [**Sau16**]).

Definition 7.3.8. — Let $U \subseteq \overline{X}$ be an open subset. We say that a function $h : U \setminus (U \cap D) \to \mathbb{C}$ has *moderate growth* on $U$ if for every point $x_0 \in D \cap U$ there exists a neighbourhood $V$ of $x_0$ and a rational function $g$ on $V$ whose poles are contained in $D \cap V$, such that for some open $W \subseteq U \cap V$ the inequality $|h(x)| \leqslant |g(x)|$ holds for all $x \in W \setminus (W \cap D)$.

Let $U \subseteq B$ be an open subset. We say that a function $h : U \setminus (U \cap \partial B) \to \mathbb{C}$ has *moderate growth* if for every point $b_0 \in \partial B \cap U$ there exists a neighbourhood $V$ of $\pi(b_0)$ and a rational function $g$ on $V$ whose poles are contained in $D \cap V$, such that for some open $W \subseteq U \cap \pi^{-1}(V)$ the inequality $|h(x)| \leqslant |g(x)|$ holds for $x \in W \setminus (W \cap \partial B)$.

7.3.9. — In the first part of the definition we could replace $V$ by $W$, hence assume $W = U \cap V$, but not so in the second part. As it is custom for meromorphic functions too, we will speak about functions of moderate growth on open subsets $U \subseteq X$ or $U \subseteq B$, when we really mean functions on $U \setminus (U \cap D)$ or $U \setminus (U \cap \partial B)$. Meromorphic functions on $X$ with poles on $D$ have moderate growth. Finite sums and products of functions of moderately growing functions grow moderately. Having moderate growth is a local condition, hence the presheaves on $X$ and on $B$ given by

$$\begin{aligned} \mathcal{O}^{\mathrm{an}}_{X,D}(U) &:= \text{holomorphic functions on } U \setminus \partial U \text{ with moderate growth on } U \subseteq \overline{X} \\ \mathcal{O}^{\mathrm{an}}_{B,\partial B}(U) &:= \text{holomorphic functions on } U \setminus \partial U \text{ with moderate growth on } U \subseteq B \end{aligned}$$

are indeed sheaves. For any open $U \subseteq X$, holomorphic functions on $U \setminus D$ with moderate growth are in fact meromorphic functions with poles in $D$. The sheaf we call $\mathcal{O}^{\mathrm{an}}_{X,D}$ is more commonly denoted $\mathcal{O}^{\mathrm{an}}_X[*D]$.

Lemma 7.3.10. — *Let $U \subseteq X$ be an open subset. A function $h$ on $U$ has moderate growth if and only if the composite $h \circ \pi$ on $\pi^{-1}(U)$ has moderate growth. In particular, the following equality* *holds.*

$$\pi_*\mathcal{O}^{\mathrm{an}}_{B,\partial B} = \mathcal{O}^{\mathrm{an}}_{X,D}$$

PROOF. This follows form the fact that $\pi$ is a proper map. □

7.3.11. — Corollary II, 1.1.8 of [**Sab91**] states that the higher direct images $R^p\pi_*\mathcal{O}^{\mathrm{an}}_{B,\partial B}$ vanish, or in other words, that the map

$$R\pi_*\mathcal{O}^{\mathrm{an}}_{B,\partial B} \to \mathcal{O}^{\mathrm{an}}_{X,D}$$

is an isomorphism in the derived category of sheaves on $X$. Sabbah deduces this from a general Dolbeault-Grothendieck Lemma on the real blow-up $B$.

7.3.12. — We now extend the definitions of sheaves of functions with moderate growth to differential forms with moderate growth. Following Hien and Roucairol, we define the sheaf of analytic *differential $p$-forms with moderate growth* as the sheaves

$$\Omega^{\mathrm{an},p}_{X,D} := \mathcal{O}^{\mathrm{an}}_{X,D} \otimes_{\mathcal{O}^{\mathrm{an}}_X} \Omega^{\mathrm{an},p}_X \qquad \text{and} \qquad \Omega^{\mathrm{an},p}_{B,\partial B} := \mathcal{O}^{\mathrm{an}}_{B,\partial B} \otimes_{\pi^*\mathcal{O}^{\mathrm{an}}_X} \pi^*\Omega^{\mathrm{an},p}_X$$

on $X$, respectively on the real blow-up $B$. This looks more difficult than it is. On an open, say $U \subseteq B$, a section of $\Omega^{\mathrm{an},p}_{B,\partial B}$ is a finite linear combination of expressions of the form $h \otimes \omega$ or just $h\omega$, where $h$ is a holomorphic function with moderate growth on $U$, and $\omega$ is a holomorphic $p$-form defined in a neighbourhood of $\pi(U)$. The usual rules of computation apply.

These sheaves of differential forms with moderate growth come equipped with obvious differential maps, which we all denote by $d_f$. Let us give the local description of the differential $\Omega^{\mathrm{an},p}_{B,\partial B} \to \Omega^{\mathrm{an},p+1}_{B,\partial B}$. Fix a point $b \in B$, set $x = \pi(b)$, and choose local coordinate functions $x_1, \ldots, x_n$ around $x \in X$ such that $D$ is given by the equation $x_1x_2\cdots x_m = 0$ for some $0 \leqslant m \leqslant n$. If $m = 0$, then $x$ lies not on $D$ and $b$ not on the boundary $\partial B$. For a subset $I \subseteq \{1, 2, \ldots, n\}$, say with elements $i_1 < i_2 < \cdots < i_p$, set

$$dx_I := dx_{i_1}dx_{i_2}\cdots dx_{i_p}$$

so that in a neighbourhood of $x$ the $p$-forms $dx_I$ form a $\mathcal{O}^{\mathrm{an}}_X$-basis of $\Omega^{\mathrm{an},p}_X$ as $I$ runs through the subsets of $\{1, 2, \ldots, n\}$ of cardinality $p$. A moderate $p$-form $\eta$ can be written, in a sufficiently small neighbourhood of $b$, as

$$\eta = \sum_{\#I=p} u_I dx_I \tag{7.3.12.1}$$

where the coefficients $u_I$ are holomorphic functions with moderate growth. The differential of $\eta$ is given by

$$d_f(\eta) = \sum_{\#I=p} \Big(\sum_{j\notin I} \tfrac{\partial u_I}{\partial x_j} + \tfrac{\partial f}{\partial x_j} u_I\Big) dx_j dx_I$$

where the inner sum could as well run over all $j \in \{1, 2, \ldots, n\}$, only that the terms with $j \in I$ are zero. The description of the differential for smooth forms on $B$ is similar, only that this time we need to choose $2n$ real coordinate functions on around $x = \pi(b)$.

7.3.13. — The real blow-up $B$ comes with the function $f : B \to \widetilde{\mathbb{P}}^1$. We denote by $\partial^+\widetilde{\mathbb{P}}^1 \subseteq \widetilde{\mathbb{P}}^1$ the half-circle of nonnegative real part, and set $\partial^+ B := f^{-1}(\partial^+\widetilde{\mathbb{P}}^1)$. Let us denote by $\kappa$ the inclusion of the open complement of $\partial^+ B$ into $B$, so that cohomology on $B$ of the constructible sheaf

$$\underline{\mathbb{C}}_{[B,\partial^+ B]} := \kappa_! \kappa^* \underline{\mathbb{C}}_B$$

is the cohomology of the pair of spaces $[B, \partial^+ B]$. The cohomology of the pair is indeed the rapid decay cohomology of $(X, f)$ with complex coefficients. We define a morphism of sheaves

$$\varepsilon : \underline{\mathbb{C}}_{[B,\partial^+ B]} \to \mathcal{O}^{\mathrm{an}}_{B,\partial B}$$

on $B$ as follows: Given a connected open subset $U$ of $B$, we have either $U \cap \partial^+ B \neq \varnothing$ in which case $\underline{\mathbb{C}}_{[B,\partial^+ B]}(U) = 0$, or we have $U \cap \partial^+ B = \varnothing$ in which case $\underline{\mathbb{C}}_{[B,\partial^+ B]}(U) = \mathbb{C}$ and we send $\lambda \in \mathbb{C}$ to the function $x \longmapsto \lambda e^{f(x)}$ on $U \setminus \partial U$, which indeed has moderate growth.

Theorem 7.3.14 (Poincaré Lemma). — *The complex of sheaves $\Omega^{\mathrm{an},\bullet}_{B^\circ,\partial B^\circ}$ is a resolution of the sheaf $\underline{\mathbb{C}}_{[B^\circ,\partial B^\circ]}$ via the map $\varepsilon$.*

Proof. We must show that the complex of sheaves and morphisms of sheaves on $B^\circ$

$$0 \longrightarrow \underline{\mathbb{C}}_{[B^\circ,\partial B^\circ]} \xrightarrow{\varepsilon} \mathcal{O}^{\mathrm{an}}_{B^\circ,\partial B^\circ} \xrightarrow{d_f} \Omega^{\mathrm{an},1}_{B^\circ,\partial B^\circ} \xrightarrow{d_f} \Omega^{\mathrm{an},2}_{B^\circ,\partial B^\circ} \xrightarrow{d_f} \cdots$$

is exact. Exactness of a complex of sheaves is a local question, so we fix a point $b \in B^\circ$ and show that the corresponding sequences of stalks are exact. For notational convenience, let us introduce for $1 \leqslant i \leqslant n$ the linear differential operator

$$Q_i(u) = \tfrac{\partial u}{\partial x_i} + \tfrac{\partial f}{\partial x_i} u$$

and for a subset $J \subseteq \{1, 2, \dots, n\}$ and $j \in J$, let us write $\mathrm{sgn}_J(j) = (-1)^{\#\{i \in J \mid i < j\}}$ so that $dx_j dx_{J\setminus\{j\}} = \mathrm{sgn}_J(j) dx_J$ holds. With these notations, the differential of a moderate $p$-form $\eta$ as in (7.3.12.1) is given by

$$d_f(\eta) = \sum_{\#J=p+1} \Big( \sum_{j\in J} \mathrm{sgn}_J(j) Q_j(u_{J\setminus\{j\}}) \Big) dx_J$$

Let $\omega \in (\mathcal{A}^{\mathrm{mod}} \otimes \pi^*\Omega^{\mathrm{an},p+1}[*D])_b$ be a germ of a moderate $(p+1)$-form with $d_f(\omega) = 0$, and let us show that $\omega = d_f(\eta)$ for some moderate $p$-form $\eta$. We can write $\omega$ as

$$\omega = \sum_{\#J=p+1} h_J dx_J \qquad\qquad 0 = d_f(\omega) = \sum_{\#K=p+2} \Big( \sum_{k\in K} \mathrm{sgn}_K(k) Q_k(h_{K\setminus\{k\}}) \Big) dx_K \tag{7.3.14.1}$$

and consider the largest integer $r \geqslant 1$ for which the implication $\{1, 2, \dots, r-1\} \cap J \neq \varnothing \implies h_J = 0$ holds. If $r = n+1$ then $\omega = 0$ and there is nothing to prove. Reasoning by induction on $r$, we only need to show that there exists a $p$-form $\eta$, say as given by (7.3.12.1), such that the coefficient of $dx_J$ in $\omega - d_f(\eta)$ is zero whenever $\{1, 2, \dots, r\} \cap J \neq \varnothing$. This amounts to solving a system of linear partial differential equations in the unknown functions $u_I$. Concretely, this system is given by

$$0 = h_J - \sum_{j\in J} \mathrm{sgn}_J(j) Q_j(u_{J\setminus\{j\}}) \tag{7.3.14.2}$$

with one equation for every subset $J \subseteq \{1,2,\dots,n\}$ with $p+1$ elements, containing at least one element $j \leqslant r$. Pick any $k \leqslant r-1$ and $J \subseteq \{1,\dots,n\}$ of cardinality $p+1$ with $h_J \neq 0$, and set $K = J \cup \{k\}$. The term in $d_f(\omega) = 0$ corresponding to $K$ just reads $Q_k(h_J) = 0$ because for any other $s \in K$ we have $k \in K \setminus \{s\}$, hence $h_{K\setminus\{s\}} = 0$ and hence $Q_s(h_{K\setminus\{s\}}) = 0$. For a similar reason, we will suppose that $u_I = 0$ as soon as $I$ contains an element $i < r$. One way of solving (7.3.14.2) is to produce for every subset $I \subseteq \{r, r+1, \dots, n\}$ a solution $u_I$ of the partial differential equation

$$(\Sigma_I): \quad \begin{cases} Q_s(u) & = 0 \quad \text{for } 1 \leqslant s < r \\ Q_r(u) & = h \end{cases} \tag{7.3.14.3}$$

with the given $h = h_{I\cup\{r\}}$, knowing that $h$ is holomorphic and has moderate growth and that the integrability condition

$$Q_s(h) = 0 \qquad \text{for } 1 \leqslant s < r \tag{7.3.14.4}$$

holds. If these solutions $u_I$ are holomorphic and have moderate growth, then the form $\eta$ as given in (7.3.12.1) has the desired properties and the proof is done. The existence of the solutions $u_I$ is precisely what Theorem 7.3.6 provides.

□

## 7.4. Proof of the comparison theorem

We keep the notations and assumptions of the previous section as presented in 7.3.7.

THEOREM 7.4.1. — *Let $X$ be a smooth affine complex variety, $Y \subseteq X$ a closed subvariety, and $f$ a regular function on $X$. The period pairing defined in* (7.0.6.1) *non-degenerate, in the sense that the induced morphism of complex vector spaces*

$$\alpha_{X,Y,f} \;:\; H^n_{\mathrm{dR}}(X,Y,f) \longrightarrow H^n_{\mathrm{rd}}(X,Y,f) \otimes_{\mathbb{Q}} \mathbb{C}$$

*is an isomorphism. Moreover, this isomorphism is functorial with respect to the morphisms (a), (b), (c) of Definition 4.2.1.*

PROPOSITION 7.4.2. — *The canonical morphism in the derived category of sheaves on $X$*

$$\Omega^{\mathrm{an},\bullet}_{X,D} = \pi_*\Omega^{\mathrm{an},\bullet}_{B,\partial B} \to \mathbb{R}\pi_*\Omega^{\mathrm{an},\bullet}_{B,\partial B}$$

*is an isomorphism in the derived category of constructible sheaves on $X$.*

PROOF. By Theorem 7.3.14, the map $\varepsilon : \underline{\mathbb{C}}_{[B,\partial^+B]} \to \Omega^{\mathrm{an},\bullet}_{B,\partial B}$ is a quasi-isomorphism, hence an isomorphism

$$\mathbb{R}\pi_*\underline{\mathbb{C}}_{[B,\partial^+B]} \xrightarrow{\cong} \mathbb{R}\pi_*\Omega^{\mathrm{an},\bullet}_{B,\partial B}$$

in the derived category of sheaves on $X$, induced by $\varepsilon$. The object $\mathbb{R}\pi_*\underline{\mathbb{C}}_{[B,\partial^+B]}$ is an object in the derived category of constructible sheaves, stratified by intersections of components of $D$. We now prove a Poincaré Lemma on $X$, providing a quasi-isomorphism

$$\mathbb{R}\pi_*\underline{\mathbb{C}}_{[B,\partial^+B]} \to \Omega^{\mathrm{an},\bullet}_{X,D}$$

inverse to the canonical map in the statement of the proposition.

. This is a local question. Let us fix a point in $X$ and choose local coordinates $x_1, \ldots, x_n$ such that $D$ is given by the vanishing of $x_1x_2\cdots x_m$ and $f$ is the function

$$f(x_1, \ldots, x_n) = \frac{1}{x_1^{e_1}\cdots x_k^{e_k}}$$

for some $0 \leqslant k \leqslant m \leqslant n$ and exponents $e_i \geqslant 0$. The fibre of $\pi : B \to X$ over $x$ is the real torus $T := \pi^{-1}(x)$ with coordinates $w_1, \ldots, w_r$, and the extension of $f$ to $B$ is given on $T$ by $f(w) = w_1^{-e_1}\cdots w_k^{-e_k}$.

The stalk at $x$ of $\mathbb{R}\pi_*\underline{\mathbb{C}}_{[B,\partial^+B]}$ is a complex of vector spaces computing the cohomology of the pair $(T, T \cap \partial^+B)$.

$$\dim H^p(T, T\cap\partial^+B) = e_1e_2\cdots e_k \cdot \binom{m-k}{p}$$

□

7.4.3. — Setup for proof

Proof of Theorem 7.4.1. Can suppose $X$ smooth, $Y$ empty. The rapid decay cohomology of $(X, f)$ with complex coefficients is the cohomology of the sheaf $\mathbb{C}_{(B,f)}$ on $B$ and we have already produced a quasi-isomorphism of complexes of sheaves

$$\varepsilon : \mathbb{C}_{[B,\partial^+B]} \xrightarrow{\cong} \Omega^{\mathrm{an},\bullet}_{B,\partial B}$$

on $B$, inducing isomorphisms of vector spaces $H^n_{\mathrm{rd}}(X, f)\otimes\mathbb{C} \cong H^n(B, \Omega^{\mathrm{an},\bullet}_{B,\partial B})$. □

CHAPTER 8

# The period realisation

In this chapter, we construct a realisation functor from $\mathbf{M}^{\exp}(k)$ to the category $\mathbf{PS}(k)$ of period structures over $k$.

## 8.1. Period structures

In this section, we introduce a tannakian formalism of *period structures* to which are associated *period algebras*, which permits us to deal abstractly with the situation where we are given vector spaces over $\mathbb{Q}$ (rapid decay cohomology) and over $k$ (de Rham cohomology) and a period isomorphism defined over $\mathbb{C}$. We fix for the whole section a subfield $k$ of $\mathbb{C}$.

Definition 8.1.1. —

(1) A *period structure* over $k$ is a triple $(V, W, \alpha)$ consisting of a finite-dimensional $\mathbb{Q}$–vector space $V$, a finite-dimensional $k$–vector space $W$, and an isomorphism of complex vector spaces $\alpha\colon V \otimes_{\mathbb{Q}} \mathbb{C} \to W \otimes_k \mathbb{C}$.

(2) A morphism of period structures $(V, W, \alpha) \to (V', W', \alpha')$ is a pair $(f_V, g_W)$ consisting of a $\mathbb{Q}$–linear map $f_V\colon V \to V'$ and a $k$–linear map $f_W\colon W \to W$ such that the following diagram of complex vector spaces commutes:

$$\begin{array}{ccc} V \otimes_{\mathbb{Q}} \mathbb{C} & \xrightarrow{f_V \otimes_{\mathbb{Q}} \mathrm{id}} & V' \otimes_{\mathbb{Q}} \mathbb{C} \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \alpha'} \\ W \otimes_k \mathbb{C} & \xrightarrow{f_W \otimes_k \mathrm{id}} & W' \otimes_k \mathbb{C}. \end{array}$$

Denote the resulting category by $\mathbf{PS}(k)$. We equip it with the evident $\mathbb{Q}$–linear monoidal structure, and regard it as a neutral $\mathbb{Q}$–linear tannakian category with the forgetful functor $(V, W, \alpha) \longmapsto V$ as fibre functor.

Definition 8.1.2. — Let $P = (V, W, \alpha)$ be a period structure, let $v_1, \ldots, v_n$ be a basis of $V$ and let $w_1, \ldots, w_n$ be a basis of $W$. Let $\underline{\alpha}$ be the matrix of $\alpha$ with respect to the bases $v_1 \otimes 1, \ldots, v_n \otimes 1$ of $V \otimes_{\mathbb{Q}} \mathbb{C}$ and $v_1 \otimes 1, \ldots, v_n \otimes 1$ of $W \otimes_k \mathbb{C}$. The *period algebra* associated with $P$ is the $k$–algebra $A$ generated by the coefficients of $\underline{\alpha}$ and $\det(\underline{\alpha})^{-1}$. The *period field* of $P$ is the fraction field of $A$.

8.1.3. — Let $P = (V, W, \alpha)$ be a period structure. We call *Galois group* of $P$ the tannakian fundamental group $G$ of the full tannakian subcategory $\langle P\rangle$ of $\mathbf{PS}(k)$ generated by $P$. It is a linear algebraic group over $\mathbb{Q}$. Let $A$ be the period algebra of $P$. There is a canonical $G_k$–torsor $T$, called *torsor of formal periods* and a canonical morphism from $\mathrm{Spec}(A)$ to $T$, which we shall construct now.

Every object of $\langle P\rangle$ can be obtained from $P$ by tensor constructions and extracting subquotiets. The category $\langle P\rangle$ comes equipped with two canonical functors: the fibre functor $\sigma$ with values in rational vector spaces given by $\sigma(V', W', \alpha') = V'$, and the other one with values in $k$–linear spaces given by $\tau(V', W', \alpha') = W'$. The group $G$ is the affine linear group scheme over $\mathbb{Q}$ which represents the following functor ([**Sza09**]):

$$G : \{\text{commutative } \mathbb{Q}\text{–algebras}\} \to \{\text{Groups}\} \qquad G(R) = \mathrm{Aut}^{\otimes}_R(\sigma \otimes R)$$

To give an element of $G(R)$ is to give for every period structure $(V', W', \alpha')$ in $\langle P\rangle$ an $R$–linear automorphism $g_{(V',W',\alpha')} : V' \otimes R \to V' \otimes R$, and these automorphisms are required to be compatible with morphisms of period structures and tensor products. In particular $g_{(V,W,\alpha)}$ determines $g_{(V',W',\alpha')}$ for every other object $(V', W', \alpha')$ of $\langle P\rangle$, hence $G$ can be viewed as a subgroup of $\mathrm{GL}_V$. The group $G_k = G \times_{\mathbb{Q}} k$ over $k$ is given by the "same" functor, but now viewed as a functor from $k$–algebras to groups.

Next, we wish to understand the torsor of formal periods $T$. This shall be a $G_k$–torsor (aka. principal homogeneous space), which we first describe as a functor:

$$T : \{\text{commutative } k\text{–algebras}\} \to \{\text{Sets}\} \qquad T(R) = \mathrm{Isom}^{\otimes}_R(\tau \otimes_{\mathbb{Q}} R, \sigma \otimes_k R)$$

The group $G_k(R)$ acts simply transitively on the set $T(R)$ on the left, for as long as $T(R)$ is not empty. Notice that $T(\mathbb{C})$ contains a canonical element given by $(V, W, \alpha) \longmapsto \alpha$, hence $T$ is not the empty functor. By [Milne, Etale cohomology, Theorem III.4.3a - representability of torsors under affine group schemes], the functor $T$ is representable by an affine scheme of finite type $T$ over $k$.

Proposition 8.1.4. — *Let $P$ be a period structure with torsor of formal periods $T$ and period algebra $A$. There exists a canonical closed immersion of $k$–schemes $\varepsilon\colon \mathrm{Spec}(A) \to T$. Its image is the Zariski closure of $\alpha \in T(\mathbb{C})$.*

Proof. Set $T = \mathrm{Spec}(B)$. The $\mathbb{C}$–valued point $\alpha$ on $T$ corresponds to a morphism of $k$–algebras $B \to \mathbb{C}$, namely the evaluation at $\alpha$. We claim that the image in $\mathbb{C}$ of this evaluation morphism is the period algebra $A$. Once this claim is proven, we define $\varepsilon\colon \mathrm{Spec}(A) \to T$ to be the corresponding morphism of affine schemes. This morphism $\varepsilon$ is then indeed a closed immersion since $A$ is an integral ring, and its image is the Zariski closure of $\alpha \in T(\mathbb{C})$ by construction.

A regular function on $T$ is uniquely determined by a regular function on the variety of $k$–linear isomorphisms from $V \otimes_k k$ to $W$, which is affine and contains $T$ as a closed subvariety. Thus, given bases $v_1, \ldots, v_n$ of $V$ and $w_1, \ldots, w_n$ of $W$, the algebra $B$ is generated by elements $b_{ij}$ and $\det((b_{ij})_{1\leqslant i,j\leqslant n})^{-1}$. An $R$–valued point of $t \in T(R)$ is an isomorphism $t : V \otimes_{\mathbb{Q}} R \to W \otimes_k R$ and the evaluation of $b_{ij}$ at $t$ is determined by the formula

$$t(v_i \otimes 1) = \sum_{j=1}^{n} w_j \otimes b_{ij}(t)$$

which in the case $R = \mathbb{C}$ and $t = \alpha$ shows the desired equality. □

8.1.5. — Here is an alternative, equivalent definition of $\varepsilon$ as a morphism of functors $\varepsilon\colon \operatorname{Spec}(A) \to T$. Fix bases of $V$ and $W$ as in the proof of the proposition. For every morphism of $k$–algebras $f\colon A \to R$, we obtain an $R$–linear isomorphism $V \otimes_{\mathbb{Q}} R \to W \otimes_k R$ given by

$$\varepsilon(f)(v_i \otimes 1) = \sum_{j=1}^{n} w_i \otimes f(a_{ij})$$

which is independent of the choice of bases and defines an element of $T(R)$. If $g : A \to R$ is another algebra morphism, then $\varepsilon(f) = \varepsilon(g)$ implies $f(a_{ij}) = g(a_{ij})$ for all $1 \leqslant i, j \leqslant n$, hence $f = g$. Therefore $\varepsilon$ is injective.

Definition 8.1.6. — Let $P$ be a period structure with torsor of formal periods $T$ and period algebra $A$. We say that $P$ is *normal* if the canonical morphism $\operatorname{Spec}(A) \to T$ is an isomorphism.

8.1.7. — If two period structures $P$ and $P'$ generate the same tannakian subcategory of $\mathbf{PS}(k)$, then $P$ and $P'$ have canonically isomorphic Galois groups and period torsors, and their period algebras are equal. Hence $P$ is normal if and only if $P'$ is. It is not hard to show that any substructure, quotient structure or tensor construction of a normal period structure is again normal. However, the sum of two normal structures might not be normal.

Example 8.1.8. — It is not hard to give examples of normal and non-normal period structures.

(1) Consider the case $k = \mathbb{Q}$ and $V = W = \mathbb{Q}$, so that $\alpha$ is just a complex number. The period structure $(\mathbb{Q}, \mathbb{Q}, \alpha)$ is normal if and only if $\alpha$ is transcendental or a rational multiple of a root of unity or a rational multiple of the square root of a rational number.

(2) Let $(V, W, \alpha)$ be a normal period structure with period algebra $A$. Elements of $A$ which are algebraic over $k$ form a normal field extension of $k$.

(3) Let $F$ be a finite field extension of $k$. Let $V$ be the rational vector space with basis the complex embeddings $\varphi_1, \ldots, \varphi_n$ of $F$, let $w_1 \ldots, w_n$ be a $k$–basis of $W = F$ and set

$$\alpha(\varphi_i \otimes 1) = \sum_{j=1}^{n} w_j \otimes \varphi_i(w_j)\,.$$

The period structure $(V, W, \alpha)$ is normal. The period algebra of $(V, W, \alpha)$ is the normalisation of $F$ in $\mathbb{C}$.

Proposition 8.1.9. — *Let $P_0$ be normal period structure. The following holds:*

*(0) The unit structure $(\mathbb{Q}, k, 1)$ is normal.*OK

*(1) Every substructure, quotient and tensor construction of $P_0$ is normal.OK*

Proof. Statemenent (0) is trivial. To prove statement (1), pick any substructure $P = (V, W, \alpha)$ of $P_0$. The Galois group $G$ of $P$ is a quotiemt of the Galois group $G_0$ of $P_0$, and there is a corresponding surjective morphism of formal period torsors $T_0 \to T$. On $R$–points, the map $T_0(R) \to T(R)$ is given by restriction. The period algebra $A$ of $P$ is contained in the period algebra $A_0$ of $P_0$, and the diagram

$$\begin{array}{ccc} \mathrm{Spec}(A_0) & \longrightarrow & \mathrm{Spec}(A) \\ \Big\downarrow{\scriptstyle \varepsilon_0} & & \Big\downarrow{\scriptstyle \varepsilon} \\ T_0 & \twoheadrightarrow & T \end{array}$$

commutes, hence $\varepsilon\colon \mathrm{Spec}(A) \to T$ is surjective, hence an isomorphism. The same argument settles the case where $P$ is a quotient or a tensor construction of $P_0$, or in fact any object in the tannakian category $\langle P_0 \rangle$ generated by $P_0$, hence statement (1) is proven. □

8.1.10. — To every object $[X, Y, f, n, i]$ in $\mathrm{Q}^{\exp}(k)$ we associate the period structure

$$\big(H^n([X, Y, f], \mathbb{Q})(i), H^n_{\mathrm{dR}}([X, Y, f]/k), \alpha\big)$$

where $\alpha$ is the comparison isomorphism... We obtain this way a quiver representation of $\mathrm{Q}^{\exp}(k)$ in the category of period structures.

## 8.2. The period realisation and the de Rham realisation

In this section, we construct a fibre functor

$$\mathrm{R}_{\mathrm{dR}}\colon \mathbf{M}^{\exp}(k) \longrightarrow \mathbf{Vec}_k,$$

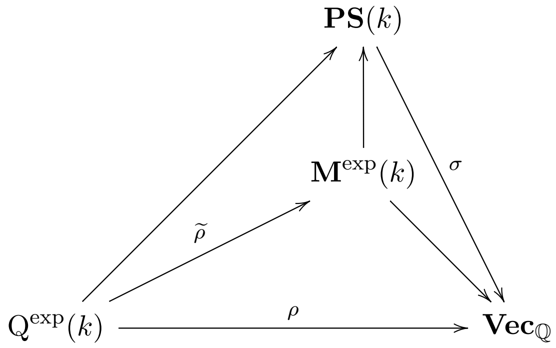
which we call the *de Rham realisation*, as well as a canonical isomorphism $\mathrm{R}_{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{C} \simeq \mathrm{R}_{\mathrm{dR}} \otimes_k \mathbb{C}$.

The following theorem is nothing else but a restatement of Theorem 7.4.1.

Theorem 8.2.1. — *Let $\mathrm{Q}^{\exp}(k)$ denote the quiver of There exists a canonical isomorphism of quiver representations*

$$\mathrm{comp}_{B,\mathrm{dR}}\colon \mathrm{R}_{\mathrm{dR}} \otimes_k \mathbb{C} \xrightarrow{\sim} \mathrm{R}_B \otimes_{\mathbb{Q}} \mathbb{C}.$$

Definition 8.2.2. — The period realisation functor $\mathrm{R}_{\mathbf{PS}} : \mathbf{M}^{\exp}(k) \to \mathbf{PS}(k)$ is the unique functor which renders the following diagram commutative.




The *de Rham realisation* $\mathrm{R}_{\mathrm{dR}} : \mathbf{M}^{\exp}(k) \to \mathbf{Vec}_k$ is the composite of the period realisation functor and the forgetful functor $\mathbf{PS}(k) \to \mathbf{Vec}_k$.

Conjecture 8.2.3 (Exponential period conjecture). — *The period realisation functor is fully faithful. For every motive $M$, the associated period structure $\mathrm{R}_{\mathbf{PS}}(M)$ is normal.*

8.2.4. — The period conjecture 8.2.3 consists of two statements. The full faithfulness of the period realisation functor is sometimes referred to as *formal period conjecture*. Given a motive $M$ with period structure $P := \mathrm{R}_{\mathbf{PS}}(M)$ and writing $G_M$ and $G_P$ are the tannakian fundamental groups of $M$ and $P$, the formal part of the period conjecture states that the inclusion of algebraic groups

$$G_P \xrightarrow{\subseteq} G_M$$

is an equality. This equality of groups can be verified in many examples, often by some trickery with algebraic groups and very limited information about the involved periods. The second statement of Conjecture 8.2.3 is that the period structure $P$ of $M$ is normal.

This leads to the following numerical variant of the exponential period conjecture.

Conjecture 8.2.5. — *Let $M$ be an exponential motive over $\overline{\mathbb{Q}}$ with motivic Galois group $G_M$. Then*

$$\operatorname{trdeg} \overline{\mathbb{Q}}(\textit{periods of } M) = \dim G_M.$$

## 8.3. Comparison with the Kontsevich-Zagier definition

At the end of [**KZ01**], Kontsevich and Zagier introduced the following definition of exponential periods. Compare both definitions.

## 8.4. Motivic exponential periods

PROPOSITION 8.4.1. — *The scheme of tensor isomorphisms* $\underline{\mathrm{Isom}}^{\otimes}(R_{\mathrm{dR}}, R_B)$ *forms a torsor under the motivic exponential Galois group.*

DEFINITION 8.4.2. — The ring of *motivic exponential periods* is

$$\mathcal{P}^{\mathfrak{m}}_{\exp} = \mathcal{O}(\underline{\mathrm{Isom}}^{\otimes}(\mathrm{R}_{\mathrm{dR}}, \mathrm{R}_B)). \tag{8.4.2.1}$$

A typical object of $\mathcal{P}^{\mathfrak{m}}_{\exp}$ is a triple $[M, \omega, \gamma]^{\mathfrak{m}}$ consisting of an exponential motive $M$ in $\mathbf{M}^{\exp}(\mathbb{Q})$, together with elements $\omega \in \mathrm{R}_{\mathrm{dR}}(M)$ and $\sigma \in \mathrm{R}_B(M)^{\vee}$. Such a triple is called a *matrix coefficient* and defines a regular function on the scheme of tensor isomorphisms via

$$\underline{\mathrm{Isom}}^{\otimes}(R_{\mathrm{dR}}, R_B) \to \mathbb{A}^1_{\mathbb{Q}}, \quad \varphi \longmapsto \langle \varphi(\omega), \sigma \rangle.$$

Indeed, one can show that $\mathcal{P}^{\mathfrak{m}}_{\exp}$ is the $\mathbb{Q}$-algebra generated by the matrix coefficients $[M, \omega, \sigma]^{\mathfrak{m}}$ modulo the following two relations:

(i) Bilinearity: for all $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{Q}$:

$$[M, \lambda_1\omega_1 + \lambda_2\omega_2, \sigma]^{\mathfrak{m}} = \lambda_1[M, \omega_1, \sigma]^{\mathfrak{m}} + \lambda_2[M, \omega_2, \sigma]^{\mathfrak{m}},$$
$$[M, \omega, \mu_1\sigma_1 + \mu_2\sigma_2]^{\mathfrak{m}} = \mu_1[M, \omega, \sigma_1]^{\mathfrak{m}} + \mu_2[M, \omega, \sigma_2]^{\mathfrak{m}}$$

(ii) *Functoriality*: if $f\colon M_1 \to M_2$ is a morphism in $\mathbf{M}^{\exp}(\mathbb{Q})$ such that $\omega_2 = R_{\mathrm{dR}}(f)(\omega_1)$ and $\sigma_1 = R_B(f)^{\vee}(\sigma_2)$, then

$$[M_1, \omega_1, \sigma_1]^{\mathfrak{m}} = [M_2, \omega_2, \sigma_2]^{\mathfrak{m}}.$$

The product is defined as

$$[M_1, \omega_1, \sigma_1]^{\mathfrak{m}}[M_2, \omega_2, \sigma_2]^{\mathfrak{m}} = [M_1 \otimes M_2, \omega_1 \otimes \omega_2, \sigma_1 \otimes \sigma_2]^{\mathfrak{m}}.$$

Evaluation at *comp* $\in \mathcal{P}^{\mathfrak{m}}_{\exp}(\mathbb{C})$ yields a map

$$\mathrm{per}\colon \mathcal{P}^{\mathfrak{m}}_{\exp} \longrightarrow \mathbb{C}.$$

The main reason to consider motivic exponential periods is that they come with a new structure, invisible at the level of numbers:

$$\Delta\colon \mathcal{P}^{\mathfrak{m}}_{\exp} \longrightarrow \mathcal{P}^{\mathfrak{m}}_{\exp} \otimes_{\mathbb{Q}} \mathcal{O}(G). \tag{8.4.2.2}$$

Let $e_1, \ldots, e_n$ be a basis of $\mathrm{R}_B(M)$. Then:

$$\Delta[M, \omega, \gamma]^{\mathfrak{m}} = \sum_{i=1}^{n} [M, \omega, e_i^{\vee}] \otimes [M, e_i, \gamma]. \tag{8.4.2.3}$$

CHAPTER 9

# The $\mathscr{D}$-module realisation

In this chapter,

## 9.1. Prolegomena on $\mathscr{D}$-modules

Six operations formalism: $f_+$, $f^+$ etc.

Introduce regular singular holonomic $\mathscr{D}$-modules

Introduce the de Rham functor

Theorem 9.1.1 (Riemann-Hilbert correspondence). — *Let $X$ be a smooth complex algebraic variety. The de Rham functor induces an equivalence of categories*

$$\mathrm{DR}_X \colon \mathrm{Mod}_{\mathrm{rh}}(\mathscr{D}_X) \longrightarrow \mathrm{Perv}(X(\mathbb{C}), \mathbb{C}).$$

## 9.2. Holonomic $\mathscr{D}$-modules on the affine line

Definition 9.2.1. — The *Weyl algebra* $\mathbb{C}[x]\langle\partial\rangle$.

In these concrete terms, holonomic means that every element of $\mathscr{M}$ is annhilated by a non-zero operator in $\mathbb{C}[x]\langle\partial\rangle$.

9.2.2 (Fourier transform). — Let $\widehat{\mathbb{A}}^1 = \operatorname{Spec} k[y]$ and consider the diagram

$$\begin{array}{ccccc} & & \mathbb{A}^1 \times \widehat{\mathbb{A}}^1 & & \\ & \overset{p}{\swarrow} & & \overset{\widehat{p}}{\searrow} & \\ \mathbb{A}^1 & & & & \widehat{\mathbb{A}}^1 \end{array}$$

The *Fourier transform* of a $\mathscr{D}$-module $\mathscr{M}$ is defined as

$$FT(\mathscr{M}) = \widehat{p}_+(p^+M \otimes \mathcal{E}^{xy}).$$

## 9.3. The $\mathscr{D}$-module realisation

Let $\mathbf{PS}(\mathbb{A}^1)$ be the category whose objects are triples $(\mathscr{M}, C, \alpha)$ consisting of a regular holonomic $\mathscr{D}$-module on $\mathbb{A}^1_k$, a $\mathbb{Q}$-perverse sheaf $C$ on $\mathbb{A}^1(\mathbb{C})$ and an isomorphism $\alpha\colon \mathrm{DR}_{\mathbb{A}^1}(\mathscr{M}) \xrightarrow{\sim} C \otimes_{\mathbb{Q}} \mathbb{C}$.

CHAPTER 10

# The $\ell$-adic realisation

## 10.1. The preverse $\ell$-adic realisation

## 10.2. Reduction modulo $p$ via nearby fibres

Let $p$ be a prime number, $q$ a power of $p$, and $k$ a finite field with $q$ elements.

10.2.1 (Fourier transform). — Let $k = \mathbb{F}_q$ be the field with $q$ elements. To an additive character $\psi\colon \mathbb{F}_q \to \overline{\mathbb{Q}}_\ell^\times$, one associates a rank one lisse sheaf $\mathcal{L}_\psi$ on $\mathbb{A}^1_k$ called the *Artin-Schreier sheaf.* It is constructed out of the map $x \longmapsto x^q - x$, which defines a finite étale morphism $\pi\colon \mathbb{A}^1_k \to \mathbb{A}^1_k$ with Galois group $\mathbb{F}_q$. Thereofore, the étale fundamental group $\pi_1^{\text{ét}}(\mathbb{A}^1_k)$ surjects onto $\mathbb{F}_q$. Composing with the character $\psi$ gives the corresponding $\ell$-adic representation $\pi_1^{\text{ét}}(\mathbb{A}^1_k) \to \overline{\mathbb{Q}}_\ell^\times$. More geometrically, $\mathcal{L}_\psi$ is the isotypical component associated to $\psi$ in the direct sum decomposition $\pi_*\overline{\mathbb{Q}}_\ell = \bigoplus_\psi \mathcal{L}_\psi$. In particular, if $\psi$ is the trivial character, then $\mathcal{L}_\psi = \overline{\mathbb{Q}}_\ell$ is the trivial sheaf.

$$\mathrm{FT}_\psi(C) = Rp_{2*}(p_1^*C \otimes \mathcal{L}_{\psi(xy)})$$

Theorem 10.2.2 (Laumon). — *If $C$ is tamely ramified, then $\mathrm{FT}_\psi(C)$ is a lisse sheaf on $\mathbb{G}_m$.*

10.2.3 (Specialisation to characteristic $p$). —

$$\begin{array}{ccccc}
\mathbb{A}^1_{\overline{\mathbb{F}}_p} & \xrightarrow{\bar\imath} & \mathbb{A}^1_{\overline{\mathbb{Z}}_p} & \xleftarrow{\bar\jmath} & \mathbb{A}^1_{\overline{\mathbb{Q}}_p} \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle\kappa} \\
\mathbb{A}^1_{\mathbb{F}_p} & \longrightarrow & \mathbb{A}^1_{\mathbb{Z}_p} & \longleftarrow & \mathbb{A}^1_{\mathbb{Q}_p}
\end{array}$$

Definition 10.2.4. — The *nearby cycles* at $p$ is

$$R\Psi_p C = \bar\imath^* R\bar\jmath_* \kappa^* C$$

DEFINITION 10.2.5 (Sawin). — A perverse sheaf $C$ on $\mathbb{A}^1_{\mathbb{Q}}$ has *good reduction* at a prime number $p$ if the following three conditions hold:

(a) the generic rank of $R\Psi_p C$ is equal to the generic rank of $C$,
(b) the singularities of $C$ lie in $\overline{\mathbb{Z}}_p$,
(c) the inertia subgroup $I_p \subset \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ acts trivially on $R\Psi_p C$.

EXAMPLE 10.2.6. — Let $r \in \mathbb{Q}$ and let $\delta_r$ be the skyscraper sheaf supported at the point $r \in \mathbb{A}^1_{\mathbb{Q}}$. If $r = 0$, then $\delta_0$ has good reduction everywhere. If $r$ is non-zero, we write $r = a/b$ with $a$ and $b$ coprime integers. Then $\delta_r$ has good reduction at $p$ if and only if $p$ does not divide $b$.

EXAMPLE 10.2.7. — The perverse sheaf $j_!\mathcal{L}_{\chi_2}[1]$ has bad reduction at $p = 2$.

$$R\Psi_p(\pi_*\overline{\mathbb{Q}}_\ell[1]) = \overline{\mathbb{Q}}_\ell[1] \oplus R\Psi_p(j_!\mathcal{L}_{\chi_2}[1])$$

THEOREM 10.2.8 (Sawin). — *If $C_1$ and $C_2$ have good reduction at $p$, then*

$$R\Psi_p(C_1 * C_2) = R\Psi_p(C_1) * R\Psi_p(C_2).$$

From this, we immediately derive that, if $C$ has good reduction at $p$, then

$$\Pi(R\Psi_p(C)) = R\Psi_p(C) * j_!j^*\overline{\mathbb{Q}}_\ell[1] = R\Psi_p(C * j_!j^*\overline{\mathbb{Q}}_\ell[1]) = R\Psi_p(\Pi(C)).$$

Therefore, $R\Psi_p$ restricts to a functor from $\mathbf{Perv}_0(\mathbb{A}^1_{\mathbb{Q}}, \overline{\mathbb{Q}}_\ell)$ to $\mathbf{Perv}_0(\mathbb{A}^1_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)$

THEOREM 10.2.9 (Sawin). — *Let $S$ be a finite set of prime numbers including $\ell$ and let $\mathbf{CC}_S$ be the full subcategory of $\mathbf{Perv}_0(\mathbb{A}^1_{\mathbb{Q}}, \overline{\mathbb{Q}}_\ell)$ consisting of those objects with good reduction outside $S$. For each prime $p \notin S$ and each non-trivial additive character $\psi$, the functor*

$$C \longmapsto H^0(\mathbb{A}^1_{\overline{\mathbb{F}}_p}, R\Psi_p C \otimes \mathcal{L}_\psi)$$

*is a fibre functor $\mathbf{CC}_S \to \mathbf{Vec}_{\overline{\mathbb{Q}}_\ell}$. The Frobenius at $p$ is an automorphism of this functor.*

## 10.3. $L$-functions of exponential motives

CHAPTER 11

# Exponential Hodge theory

In this chapter, we construct a Hodge realisation functor from the category of exponential motives to a subcategory of mixed Hodge modules over the complex affine line—parallel to $\mathbf{Perv}_0$—that Kontsevich calls *exponential mixed Hodge structures.* Throughout, "Hodge structure" means rational mixed Hodge structure. We always suppose them to be graded polarisable, that is, each pure subquotient admits a polarisation. We denote the category of Hodge structures by **MHS**. It is a $\mathbb{Q}$-linear neutral tannakian category, with respect to the forgetful functor

$$f\colon \mathbf{MHS} \to \mathbf{Vec}_{\mathbb{Q}}.$$

## 11.1. Reminder on mixed Hodge modules

The theory of Hodge modules is a long story – we will recite here a few essential properties of categories of mixed Hodge modules, and give a brief description of their construction. For a more thorough introduction see [**Sch14, SS16**].

Definition 11.1.1. — Let $X$ be a complex algebraic variety. A *pre-mixed Hodge module* on $X$ consists of the following data:

- A rational perverse sheaf $L$, together with an increasing filtration $W_\bullet L$ by perverse subsheaves.
- A regular holonomic $\mathscr{D}_X$-module $\mathcal{M}$, together with an increasing filtration $W_\bullet\mathcal{M}$ and a good filtration $F^\bullet\mathcal{M}$.
- An isomorphism $\alpha\colon DR(\mathcal{M}) \simeq L \otimes_{\mathbb{Q}} \mathbb{C}$ under which $W_\bullet\mathcal{M}$ corresponds to $W_\bullet L \otimes_{\mathbb{Q}} \mathbb{C}$.

Pre-mixed Hodge modules form a category and mixed Hodge modules are defined inductivtely as a subcategory of them.

11.1.2. — For every complex algebraic variety $X$, there is an abelian category $\mathbf{MHM}(X)$ of mixed Hodge modules on $X$, and a functor

$$\mathrm{rat} : \mathbf{MHM}(X) \to \mathbf{Perv}(X)$$

which is exact and faithful, so we may look at mixed Hodge modules as perverse sheaves with extra data, though the functor rat is not essentially surjective. Categories of mixed Hodge modules (or better: their bounded derived categories) enjoy a six functors formalism, which is compatible with the functor rat. If $X$ is a point, then the category of mixed Hodge modules on $X$ is the category of mixed, graded polarisable Hodge structures.

11.1.3. — Let $X$ be a smooth, connected algebraic variety of dimension $n$, and let $V$ be a variation of mixed Hodge structures on $X$. There is a mixed Hodge module on $X$ naturally associated with $V$, which we shall denote by $V[n]$. As the notation suggests, its underlying perverse sheaf is the local system underlying $V$ shifted to degree $-n$.

Mixed Hodge modules come with a functorial, exact weight filtration.

A formal consequence of the six functors formalism for mixed Hodge modules is that we can define additive convolution on $\mathbf{MHM}(\mathbb{C})$ as we did for perverse sheaves in 2.4.1.

When $X$ is a point, $\mathbf{MHM}(X)$ is nothing but the category of mixed Hodge structures (recall the proviso that they are assumed to be graded polarisable).

## 11.2. Exponential mixed Hodge structures

Definition 11.2.1 (Kontsevich–Soibelman). — An *exponential mixed Hodge structure* is a mixed Hodge module on the complex affine line $\mathbb{C}$ whose underlying perverse sheaf belongs to $\mathbf{Perv}_0$. We denote the corresponding full subcategory by **EMHS**.

Example 11.2.2. — Of particular interest are the exponential mixed Hodge structures

$$E(s) = j(s)_! j(s)^* \pi^* \underline{\mathbb{Q}}[1],$$

where $\Pi\colon \mathbb{A}^1_{\mathbb{C}} \to \operatorname{Spec} \mathbb{C}$ is the structure morphism, $j(s)\colon \mathbb{C} \setminus \{s\} \to \mathbb{C}$ the inclusion, and $\mathbb{Q} = \mathbb{Q}(0)$ stands for the one-dimensional Hodge structure of weight 0, regarded as a Hodge module on the point. The perverse sheaf underlying $E(0)$ was introduced under the same name in Example 2.3.4.

The inclusion of **EMHS** into $\mathbf{MHM}(\mathbb{C})$ admits as a left adjoint the exact idempotent functor

$$\begin{aligned} \Pi\colon \mathbf{MHM}(\mathbb{C}) &\longrightarrow \mathbf{EMHS} \\ M &\longmapsto M * E(0). \end{aligned}$$

Definition 11.2.3. — We call *canonical* the functor $\mathbf{MHS} \to \mathbf{EMHS}$ which sends $H$ to the exponential mixed Hodge structure $\Pi(i_* H)$, where $i\colon \{0\} \hookrightarrow \mathbb{C}$ is the inclusion. Explicitly,

$$\Pi(i_* H) = j_! j^* \pi^* H[1].$$

Observe that $\pi(i_* H)$ has singularities only at 0 and trivial monodromy.

Lemma 11.2.4. — *The canonical functor* **MHS** $\to$ **EMHS** *is fully faithful, and its essential image is stable under taking quotients and subobjects.*

Proof. The canonical functor $\iota : \mathbf{MHS} \to \mathbf{EMHS}$ is exact and faithful, because the functors $j_!$, $j^*$ and $\pi^*$ are so. To check that the canonical functor is full, let $V$ and $W$ be Hodge structures, and let

$$f\colon j_!j^*\pi^*V[1](-1) \to j_!j^*\pi^*W[1](-1)$$

be a morphism of Hodge modules. The perverse sheaf underlying $j_!j^*\pi^*V[1](-1)$ is a constant local system on $\mathbb{C}^*$ given by the rational vector space underlying $V(-1)$ in degree $-1$, and its fibre over $0 \in \mathbb{C}$ is zero. The same holds for $W$. Therefore, if $f$ induces the zero morphism on the fibre over any $z \neq 0$, then $f$ is the zero morphism. Let $i_1 : \{1\} \to \mathbb{C}$ be the inclusion. The fibre of $f$ over 1 is the morphism

$$i_1^*(f) : i_1^*j^*\pi^*V[1](-1) \to i_1^*j_!j^*\pi^*W[1](-1)$$

induced by $f$. The fibre $i_1^*j^*\pi^*V[1](-1)$ is the Hodge structure $V(-1)$ put in degree $-1$. After twisting and shifting we obtain thus a morphism of Hodge structures $f_1 : V \to W$. The difference $f - \iota(f_1)$ is then a morphism of Hodge modules and its fibre over 1 is zero, hence $f = \iota(f_1)$. Let us now check that the essential image of the canonical functor is stable under taking subobjects. Let $V$ be a Hodge structure, and let $M \subseteq j_!j^*\pi^*V[1](-1)$ be a subobject of in the category **EMHS**. Applying the left exact functor $\pi_*j_*j^*(-)[-1](1)$ we obtain a subobject

$$W := \pi_*j_*j^*M[-1](1) \subseteq \pi_*j_*j^*\pi^*V = V$$

in the category of Hodge structures. Applying $j^*\pi^*(-)[1](-1)$, using adjunction and applying $j_!$ yields a morphism

$$j_!j^*\pi^*W[1](-1) \to M \subseteq j_!j^*\pi^*V[1](-1)$$

and we need to show that the morphism of Hodge modules $j_!j^*\pi^*W[1](-1) \to M$ is an isomorphism. This is indeed the case, since the morphism of underlying perverse sheaves one obtains by applying the functor rat is an isomorphism. This shows that the essential image of the canonical functor is stable under taking subobjects, hence also under taking quotients. □

Remark 11.2.5. — Contrary to what is claimed in [**KS11**, p.262], the image of the canonical functor does not form a Serre subcategory of **EMHS**, *i.e.* is not stable under extension. Here is an example. Every graded polarisable variation of mixed Hodge structures $V$ on $\mathbb{C}^*$ determines a mixed Hodge module $V[1]$ on $\mathbb{C}^*$ with the evident underlying perverse sheaf. For example we may consider the variation of mixed Hodge structure whose fibre over $z \in \mathbb{C}$ is the Hodge realisation of the 1-motive $[\mathbb{Z} \xrightarrow{u} \mathbb{C}^*]$ given by $u(1) = z$. This variation $V$ sits in a short exact sequence

$$0 \to \mathbb{Q}(1) \to V \to \mathbb{Q} \to 0$$

and applying $j_!(-)[1]$ yields an exact sequence in **EMHS**. While the first and last term in this sequence come from Hodge structures via the canonical functor, the object in the middle does not, as the underlying perverse sheaf has a non-trivial monodromy around 0.

Proposition 11.2.6. —

(1) *Exponential mixed Hodge structures form a* $\mathbb{Q}$*-linear tannakian category. A fibre functor is given by the composite of the forgetful functor* $\mathbf{EMHS} \to \mathbf{Perv}_0$ *and the fibre functor* $\Psi_\infty \colon \mathbf{Perv}_0 \to \mathbf{Vec}_\mathbb{Q}$.

(2) *The functors* $\mathbf{MHS} \to \mathbf{EMHS} \to \mathbf{Perv}_0$ *are functors of tannakian categories, compatible with the given fibre functors. Their composite is the trivial functor, which sends a mixed Hodge structure* $V$ *to the perverse sheaf* $j_!j^*f(V)[1]$.

PROOF. □

### 11.3. Intermezzo: Extensions of groups from the tannakian point of view

Let $F$ and $H$ be groups. By an extension of $F$ by $H$ one understands a group $G$ sitting in an exact sequence $1 \to H \to G \to F \to 1$. The problem of classifying all extensions of $F$ by $H$ is a classical problem in group theory, systematically studied by Schreier, Zassenhaus, Schur, Eilenberg, Mac Lane and many others. Two types of extensions are particularly well understood: semidirect products and extensions by abelian groups. A *semidirect product* or also *split extension* is an extension such that the quotient map $G \to F$ admits a section $F \to G$. The group $F$ acts via this section on $H$ by conjugation, and reciprocally, any action $\alpha : F \to \mathrm{Aut}(H)$ defines a split extension of $F$ by $H$ by considering on the set $G = H \times F$ the group law $(h, f)(h', f') = (h\alpha(f)(h'), ff')$. *Central extensions* are those where $H$ is contained in the centre of $G$, hence in particular is commutative. Central extensions of $F$ by $H$ up to equivalence form a commutative group $\mathrm{Ext}^1(F, H)$, with the Baer-sum as group law. This group is naturally isomorphic to the group cohomology $H^2(F, H)$, where $H$ is regarded as an $F$-module with trivial $F$-action. Given a central extension $1 \to H \to G \to F \to 1$, the corresponding cohomology class is represented by the cocycle $c : F \times F \to H$ given by

$$c(f, f') = s(f)^{-1}s(f')^{-1}s(ff')$$

where $s : F \to G$ is any map, not necessarily a group homomorphism, whose composition with the quotient map $G \to F$ is the identity on $F$. The generalisation to not necessarily central extensions of $F$ by an abelian group $H$ is not difficult. Such extensions are also classified by group cohomology $H^2(F, H)$, but now with the possibly nontrivial action of $F$ on $H$ corresponding to the conjugation action. The even more general case where $F$ is not necessarily commutative was worked out by Schreier [**Sch26**] and Eilenberg-Mac Lane [**EML47**]. It inevitably leads to noncommutative group cohomology.

More generally, one would like to classify group extensions in a topos. A complete geometric solution to this problem was given by Grothendieck and Giraud [**Gir71**], and later a cohomological interpretation was given by Breen [**Bre90**]. A new problem that arises in this generality which was not seen in the elementary case of extensions of abstract groups is that in a general topos, an extension $1 \to H \to G \to F \to 1$ defines an $H$-torsor over $F$ which need not be trivial. So, unlike in the case of abstract groups, there is not always a morphism $s : F \to G$ which splits the surjection $G \to F$.

We are interested in certain extensions of affine group schemes $1 \to H \to G \to F \to 1$, namely those where the Hopf algebra underlying $G$ is, as a coalgebra, isomorphic to the tensor product of the coalgebras associated with $H$ and $F$. In other words, we are interested in certain extensions of commutative group objects in the category of not necessarily commutative coalgebras. Such extensions arise naturally when one tries to turn the vanishing cyles functor for exponential Hodge-structures into a tensor functor. Indeed, this vanishing cyles functor takes values in the category of what Scherk and Steenbrink call $\widehat{\mu}$-Hodge structure in [**SS85**], that is, mixed Hodge structures with an automorphism of finite order. The category $\mathbf{MHS}^{\widehat{\mu}}$ of $\widehat{\mu}$-Hodge structures comes equipped with a symmetric tensor product, not the obvious one, which turns it into a tannakian category. The tannakian fundamental group sits in an extension

$$0 \to \widehat{\mathbb{Z}} \to \pi_1(\mathbf{MHS}^{\widehat{\mu}}) \to \pi_1(\mathbf{MHS}) \to 1$$

which is exactly of the nature described above: as an abelian category $\mathbf{MHS}^{\widehat{\mu}}$ is the obvious thing, morphisms are morphisms of Hodge structures compatible with the automorphisms, so the coalgebra underlying the affine group scheme $\pi_1(\mathbf{MHS}^{\widehat{\mu}})$ is the tautological one. The commutative multiplication turning this coalgebra into a commutative Hopf algebra corresponds to the special tensor product we are considering.

The plan for this section is as follows: after fixing conventions, we start by describing extensions of group schemes in terms of tannakian categories and in terms of Hopf algebras. That done, we translate classical constructions from group theory such as semidirect products and the classification of extensions, in particular commutative extensions, by group cohomology into the language of coalgebras. In particular, we show how to use 2-cocycles to describe extensions of Hopf algebras.

11.3.1. — We fix a field $K$ of characteristic zero, which in all applications will be one of the typical coefficient fields $\mathbb{Q}$ or $\mathbb{Q}_\ell$, and convene that all tannakian categories under consideration are neutral $K$-linear tannakian categories, each one equipped with a fixed fibre functor to the category **Vec** of finite-dimensional vector spaces over $K$. Algebras are understood to be algebras over $K$, and schemes are accordingly defined over $K$.

PROPOSITION 11.3.2. — *A sequence of of affine group schemes $H \xrightarrow{i} G \xrightarrow{p} F$ over $\mathbb{Q}$ satisfying $p \circ i = 1$ is exact if the following two conditions are satisfied.*

*(1) For every representation $V$ of $G$, the equality $V^H = V^{\ker(p)}$ holds.*
*(2) Every one dimensional representation of $H$ which is obtained as a subquotient of some representation of $G$ can be obtained from a one dimensional representation of $G$. In other words, the restriction map $\mathrm{Hom}(G, \mathbb{G}_m) \to \mathrm{Hom}(i(H), \mathbb{G}_m)$ is surjective.*

PROOF. Let us write $N$ for the kernel of $p$, and suppose without loss of generality $H$, $G$ and $F$ are linear groups and that $H$ is a subgroup of $N \subseteq G$ via the inclusion $i$. We can deduce from condition (2) that for every representation $V$ of $G$, the equality

$$\mathbb{P}(V)^H = \mathbb{P}(V)^N \tag{11.3.2.1}$$

holds. Indeed, a line $\langle v\rangle$ in $V$ which is stable under $H$ corresponds to a character $\chi : H \to \mathbb{G}_m$. By hypothesis, we can extend this character to $\chi : G \to \mathbb{G}_m$. Let $\mathbb{Q}(\chi^{-1})$ be the one dimensional representation of $G$ with character $\chi^{-1}$. Then, $v \otimes 1 \in V \otimes \mathbb{Q}(\chi^{-1})$ fixed by $H$, hence by $N$. It follows that the line $\langle v\rangle$ is also stable under $N$. That the equality $H = N$ follows from (11.3.2.1) is an observation of dos Santos [**dS12**, Lemma 4.2 and 4.3]. The argument goes as follows: The quotient $G/H$ is a quasiprojective algebraic variety with $G$-action, $G$ acting by left translation on right cosets. By Chevalley's theorem, there exists a representation $V$ of $G$ and a $G$-equivariant immersion $\alpha : G/H \to \mathbb{P}V$. The point $\alpha(1) \in \mathbb{P}V$ is fixed by $H$, hence by $N$. This means that the equality $NH = H$ holds in $G$, hence $H = N$. □

Proposition 11.3.3. — *Let $K \to A \xrightarrow{p} E \xrightarrow{i} B \to K$ be morphisms of commutative Hopf algebras. The corresponding sequence of affine group schemes $1 \to \operatorname{Spec} B \to \operatorname{Spec} E \to \operatorname{Spec} A \to 1$ is exact if and only if the morphism $p\colon A \to E$ is injective, $i : E \to B$ is surjective and*

$$\ker(i) = E.i(A^+)$$

*where $A^+ = \ker(\varepsilon_A : A \to K)$ is the augmentation ideal of $A$.*

11.3.4. — Let $A$ and $B$ be commutative Hopf algebras and set $F = \operatorname{Spec} A$ and $H = \operatorname{Spec} B$. By an extension of $B$ by $A$ we understand a sequence of (not necessarily commutative) Hopf algebras

$$K \to A \xrightarrow{p} E \xrightarrow{i} B \to K$$

where $p\colon A \to E$ is injective, $i : E \to B$ is surjective and $\ker(i) = E.i(A^+)$, up to the usual notion of equivalence. Commutative extensions, that means those where the multiplication on $E$ is commutative, are in one to one corespondence with extensions of the group scheme $F$ by the group scheme $H$. Let us denote by $\mathrm{EXT}(B, A)$ the set of all (equivalence classes of) extensions of $B$ by $A$ and by

$$\mathrm{CEXT}(B, A) \subseteq \mathrm{EXT}(B, A)$$

the subset of commutative extensions. This are just a pointed sets, with the trivial extension $A \otimes B$ as distinguished element. Every commutative extension $E$ of $B$ by $A$ defines a $H$-torsor $G = \operatorname{Spec} E$ over the scheme $F$, corresponding to an element $t_G \in H^1_{\mathrm{fppf}}(F, H)$. If $G$ is trivial as an $H$-torsor, or in other words if $t_G = 0$, then $G$ is isomorphic as a scheme to $F \times H$. Let us denote by

$$\mathrm{EXT}_m(A, B) \subseteq \mathrm{CEXT}(A, B)$$

the subset of $\mathrm{EXT}(A, B)$ consisting of those extensions whose underlying algebra is $A \otimes B$ with the commutative multiplication $m_A \otimes m_B$, obtained from the multiplication $m_A$ on $A$ and $m_B$ on $B$. To give an element of $\mathrm{EXT}_m(A, B)$ is to give a group structure on the scheme $H \times F$ which is compatible with the inclusion $H \to H \times F$ and the projection $H \times F \to F$, or else, a comultiplication on the commutative algebra $(A \otimes B, m_A \otimes m_B)$ compatible with the morphisms $A \otimes B \to A$ and $B \to A \otimes B$. The following bijection is tautological.

$$\mathrm{EXT}_m(A,B) \xleftrightarrow{\cong} \left\{ \begin{matrix} \text{Comultiplications on the algebra } (A\otimes B, m_A \otimes m_B) \text{ which are} \\ \text{compatible with the morphisms } A\otimes B \to A \text{ and } B \to A \otimes B. \end{matrix} \right\}$$

Instead of considering extensions with fixed underlying scheme $H \times F$, that is, keeping the algebra structure $m_A \otimes m_B$ on $A \otimes B$ and modifying the comultiplication, we can also consider extensions which arise by keeping the coalgebra structure $\mu_A \otimes \mu_B$ on $A \otimes B$ and letting the algebra structure vary. Let us denote by

$$\mathrm{EXT}_\mu(A,B) \subseteq \mathrm{EXT}(A,B)$$

the subset consisting of those extensions whose underlying coalgebra is $(A \otimes B, \mu_A \otimes \mu_B)$. The following bijection is tautological.

$$\mathrm{EXT}_\mu(A,B) \xleftrightarrow{\cong} \left\{ \begin{matrix} \text{Multiplications on the coalgebra } (A\otimes B, \mu_A \otimes \mu_B) \text{ which are} \\ \text{compatible with the morphisms } A\otimes B \to A \text{ and } B \to A \otimes B. \end{matrix} \right\}$$

In categorical terms, this means we consider $\mathbf{Rep}(H\times F) = \mathbf{Comod}(A \otimes B)$ as an abelian category, and seek to modify the tensor product on it. The situation is not completely symmetric, since in our setup we require $A$ and $B$ to be commutative, but do not require them to be cocommutative. Let

$$\mathrm{CEXT}_\mu(A,B) \subseteq \mathrm{CEXT}(A,B)$$

be the subset of commutative extensions of $B$ by $A$ with underlying coalgebra $(A \otimes B, \mu_A \otimes \mu_B)$. Again we have a tautological bijection.

$$\mathrm{CEXT}_\mu(A,B) \xleftrightarrow{\cong} \left\{ \begin{matrix} \text{Commutative multiplications on the coalgebra } (A\otimes B, \mu_A \otimes \mu_B) \text{ which} \\ \text{are compatible with the morphisms } A\otimes B \to A \text{ and } B \to A \otimes B. \end{matrix} \right\}$$

Any such commutative extension of Hopf algebras gives rise to an extension of affine group schemes $1 \to H \to G \to F \to 1$. If the corresponding torsor class $t_G \in H^1_{\mathrm{fppf}}(F,H)$ is zero, then the multiplication on $A \otimes B$ is $m_A \otimes m_B$ and the comultiplication is $\mu_A \otimes \mu_B$, so the extension is trivial. In other words, the map of pointed sets

$$\mathrm{CEXT}_\mu(A,B) \to H^1_{\mathrm{fppf}}(F,H)$$

has trivial kernel.

Example 11.3.5. — Let $\mathbf{C}$ be the tannakian category of $\mathbb{Z}$-graded rational vector spaces, with its usual tensor product and the forgetful functor as fibre functor. Its tannakian fundamental group is the multiplicative group $\mathbb{G}_m$. Let $\mathbf{C}^{\widehat{\mu}}$ denote the category of pairs $(V,T)$ consisting of a graded vector space $V$ and a finite order automorphism $T$ of $V$ respecting the grading. The category $\mathbf{C}^{\widehat{\mu}}$ is abelian and semisimple, and as such equivalent to the category of representations of $\widehat{\mathbb{Z}} \times \mathbb{G}_m$. The simple objects are those $(V,T)$ where $V$ is pure for the given grading and has no proper $T$-invariant subspaces. If $T$ has order exactly $n$, then $V$ has dimension $\varphi(n)$ and the characteristic polynomial of $T$ is the cyclotomic polynomial $\Phi_n(X)$. Let us denote by

$$\mathbb{Q}(k,n) \qquad\qquad k \in \mathbb{Z}, \quad n \in \mathbb{Z}_{\geqslant 1},$$

the simple object $(V,T)$ where $V$ has degree $k$ and $T$ has order $n$. Simple objects of $\mathbf{C}$ are those of the form $\mathbb{Q}(k,1)$. They are of dimension 1. For $\alpha \in \mathbb{Q}$, set $V^\alpha = \ker(T - \exp(-2\pi i\alpha)) \subseteq V \otimes \mathbb{C}$, so that we have an eigenspace decomposition

$$V \otimes \mathbb{C} \quad = \bigoplus_{\alpha \in \mathbb{Q} \cap (-1,0]} V^\alpha \ .$$

Note that each $V^\alpha$ inherits a grading from $V$. We define the tensor product of two objects $(V,T)$ and $(V',T')$ of $\mathbf{C}^{\widehat{\mu}}$ by

$$(V,T) \otimes (V',T') = (V \otimes V', T \otimes T'),$$

where $V \otimes V'$ has the following grading:

$$\mathrm{gr}_k(V \otimes V') = \left( \bigoplus_{\alpha,\beta} \bigoplus_{i,j} (\mathrm{gr}_i(V^\alpha) \otimes \mathrm{gr}_j(V'^\beta)) \right) \cap (V \otimes V')$$

where, as before, the sums run over all $\alpha, \beta \in \mathbb{Q} \cap (-1,0]$ and all integres $i,j$ satisfying

$$i + j = \begin{cases} k & \text{if } \alpha = 0 \text{ or } \beta = 0, \\ k-2 & \text{if } \alpha + \beta = -1, \\ k-1 & \text{else.} \end{cases}$$

Let us see what happens with simple objects. If either $n_1$ or $n_2$ is equal to 1, say $n_2 = 1$, then we have

$$\mathbb{Q}(k_1,n_1) \otimes \mathbb{Q}(k_2,1) = \mathbb{Q}(k_1+k_2,n_1)$$

for all $k_1, k_2, n_1$. Suppose now that $n_1 \neq 1$ and $n_2 \neq 1$, and let $N$ be the least common multiple of $n_1$ and $n_2$. We have

$$\mathbb{Q}(k_1,n_1) \otimes \mathbb{Q}(k_2,n_2) = \mathbb{Q}(k_1+k_2+2,1)^{\eta(n_1,n_2,1)} \oplus \bigoplus_{d|N,\ d\neq 1} \mathbb{Q}(k_1+k_2+1,d)^{\eta(n_1,n_2,d)}$$

where $\eta(n_1,n_2,d)\varphi(d)$ is the number of pairs $(a_1,a_2) \in (\mathbb{Z}/N\mathbb{Z})^2$ where $a_1$ has order $n_1$, $a_2$ has order $n_2$ and $a_1 + a_2$ has order $d$. For example

$$\mathbb{Q}(0,100) \otimes \mathbb{Q}(0,100) = \mathbb{Q}(2,1)^{40} \oplus \mathbb{Q}(1,2)^{40} \oplus \mathbb{Q}(1,5)^{40} \oplus \mathbb{Q}(1,10)^{40} \oplus \mathbb{Q}(1,25)^{30} \oplus \mathbb{Q}(1,50)^{30}$$

which is $1600 = \varphi(100)^2 = 40\varphi(0) + 40\varphi(1) + 40\varphi(5) + 40\varphi(10) + 30\varphi(25) + 30\varphi(50)$ on the level of dimensions. If $(n_1,n_2) = 1$, then

$$\mathbb{Q}(k_1,n_1) \otimes \mathbb{Q}(k_2,n_2) = \mathbb{Q}(k_1+k_2+1,n_1n_2)$$

holds, and if $p$ is a prime, then

$$\mathbb{Q}(k_1,p) \otimes \mathbb{Q}(k_2,p) = \mathbb{Q}(k_1+k_2+1,p)^{p-2} \oplus \mathbb{Q}(k_1+k_2+2,1)^{p-1}.$$

11.3.6. — We now start adapting the theory of group extensions à la Schreier to the framework of Hopf algebras. More precisely, we replace groups with group objects in the category of $K$-coalgebras. Ultimately we are only concerned with commutative group objects, that is, commutative Hopf algebras, yet we need to start with semidirect products.

Definition 11.3.7. — Let $A$ and $B$ be Hopf algebras. An *action* of $B$ on $A$ is a linear map $\tau : B \otimes A \to A$ such that the following diagrams commute.

$$
\begin{array}{ccc}
B\otimes B\otimes A & \xrightarrow{1\otimes\tau} & B\otimes A \\
\downarrow{\scriptstyle m_B\otimes 1} & & \downarrow{\scriptstyle \tau} \\
B\otimes A & \xrightarrow{\tau} & A
\end{array}
\qquad
\begin{array}{ccc}
K\otimes A & \xrightarrow{e_B\otimes 1} & B\otimes A \\
\| & \swarrow{\scriptstyle \tau} & \\
A & &
\end{array}
\qquad
\begin{array}{ccc}
B & \xrightarrow{1\otimes e_A} & B\otimes A \\
\downarrow{\scriptstyle \varepsilon_B} & & \downarrow{\scriptstyle \tau} \\
K & \xrightarrow{e_A} & A
\end{array}
$$

$$
\begin{array}{ccccc}
B\otimes A\otimes A & \xrightarrow{\mu_B\otimes 1\otimes 1} & B\otimes B\otimes A\otimes A & \xrightarrow{|\times|} & B\otimes A\otimes B\otimes A \\
\downarrow{\scriptstyle 1\otimes m_A} & & & & \downarrow{\scriptstyle \tau\otimes\tau} \\
B\otimes A & \xrightarrow{\tau} & A & \xleftarrow{m_A} & A\otimes A
\end{array}
$$

We call *trivial action* the action defined by $\tau(b\otimes a) = \varepsilon_B(b)a$.

11.3.8. — Let $A$ and $B$ be Hopf algebras, and let $\tau : B \otimes A \to A$ be an action of $B$ on $A$. We can use $\tau$ to define a multiplication $m_\tau$ on the coalgebra $A \otimes B$ as the following composite.

$$
\begin{array}{ccccc}
A\otimes B\otimes A\otimes B \xrightarrow{1\otimes\mu_B\otimes 1\otimes 1} A\otimes B\otimes B\otimes A\otimes B & & & & \\
\downarrow{\scriptstyle ||\times|} & & & & \\
A\otimes B\otimes A\otimes B\otimes B & \xrightarrow{1\otimes\tau\otimes 1\otimes 1} & A\otimes A\otimes B\otimes B & \xrightarrow{m_A\otimes m_B} & A\otimes B
\end{array}
\tag{11.3.8.1}
$$

It is straightforward to check that the so defined map $m_\tau : A \otimes B \otimes A \otimes B \to A \otimes B$ is indeed a multiplication on $A \otimes B$, compatible with the comultiplication $\mu_A \otimes \mu_B$, so that together they combine to a Hopf algebra structure on $A \otimes B$. We call this a *semidirect product*. The trivial action induces this way the multiplication $m_A \otimes m_B$. Reciprocally, given an extension of Hopf algebras of the form

$$K \to A \xrightarrow{1\otimes e_B} (A\otimes B, m) \xrightarrow{\varepsilon_A\otimes 1} B \to K$$

where on $A \otimes B$ the comultiplication is $\mu_A \otimes \mu_B$, we obtain an action $\tau_m$ of $B$ on $A$ as follows.

$$
\begin{array}{ccccc}
B\otimes A & \xrightarrow{\mu_B\otimes 1} & B\otimes B\otimes A & & A\otimes B \xrightarrow{1\otimes\varepsilon_B} A \\
& & \downarrow{\scriptstyle \times|} & & \uparrow{\scriptstyle m} \\
& & B\otimes A\otimes B & \xrightarrow{(e_A\otimes 1)\otimes(1\otimes e_B)\otimes(e_A\otimes i_B)} & (A\otimes B)^{\otimes 3}
\end{array}
\tag{11.3.8.2}
$$

It is straightforward to check that the so defined map $\tau_m : A \otimes B \otimes A \otimes B \to A \otimes B$ is indeed an action of $B$ on $A$. If the multiplication $m = m_\tau$ is obtained from a given action $\tau : B \otimes A \to A$, so that $(A \otimes B, m_\tau)$ is a semidirect coproduct, we recover $\tau$ from $m_\tau$ - this is the content of Lemma 11.3.9. On the other hand, a multiplication $m$ on $A \otimes B$ can in general not be recovered from its induced action $\tau_m$. In particular, we notice that if the multiplication $m$ on $A \otimes B$ is commutative, then the induced action $\tau_m$ is trivial, and the trivial action induces the multiplication $m_A \otimes m_B$ on $A \otimes B$.

Lemma 11.3.9. — *Let $\tau : B\otimes A \to A$ be an action of $B$ on $A$, and let $m_\tau : (A\otimes B)^2 \to A\otimes B$ be the multiplication of the corresponding semidirect product, as defined by* (11.3.8.1). *The action of $B$ on $A$ defined by means of* (11.3.8.2) *is equal to $\tau$.*

Proof. The product $m_\tau$ is expressed by

$$m_\tau(a\otimes b\otimes a'\otimes b') = \sum_{\bullet} a\tau(b_1\otimes a')\otimes b_2 b'$$

Let us pick an element $a\otimes b$ of $A\otimes B$ and check that it gets sent to $\tau(b\otimes a)$ by the composite (11.3.8.2). The element $a\otimes b$ is mapped to

$$\sum_{\bullet}(1\otimes b_1)\otimes(a\otimes 1)\otimes(1\otimes i_B(b_2))$$

in $(A\otimes B)^3$. Multiplying the three terms together with $m_\tau$ we obtain the element

$$\sum_{\bullet\bullet}\tau(b_1\otimes a)\otimes b_2 i_B(b_3) = \sum_{\bullet}\tau(b_1\otimes a)\otimes b_2$$

of $A\otimes B$. Here we used the coassociativity of $\mu_B$ and property $b=\sum b_1 i_B(b_2)$ of the antipode. Finally, applying $1\otimes\varepsilon_B$ yields the element

$$\sum_{\bullet}\varepsilon(b_2)\tau(b_1\otimes a) = \tau(b\otimes a)$$

of $A$ as desired. In this last step, we used the property $b=\sum\varepsilon(b_2)b_1$ of the counit and bilinearity of $\tau$. ☐

11.3.10. — We call an extension $K\to A\xrightarrow{p}E\xrightarrow{i}B\to K$, where the coalgebra underlying $E$ is $(A\otimes B,\mu_A\otimes\mu_B)$, *central* if the action of $B$ on $A$ is trivial.

$$\mathrm{CEXT}_\mu(A,B)\subseteq\mathrm{ZEXT}_\mu(A,B)\subseteq\mathrm{EXT}_\mu(A,B)$$

Definition 11.3.11. — Let $A$ and $B$ be commutative Hopf algebras. A *2-cocycle* of $B$ with coefficients in $A$ (for the trivial action of $B$ on $A$) is a morphism of coalgebras $c : B\otimes B\to A$ such that the following diagram commutes

$$\begin{array}{ccccccc} B^3 & \xrightarrow{\mu_B^3} & B^6 & \xrightarrow{\varepsilon_B\otimes 1\otimes 1\otimes 1\otimes m_B} & B^4 & \xrightarrow{c\otimes c} & A^2 \\ \Big\downarrow{\scriptstyle\mu_B^3} & & & & & & \Big\downarrow{\scriptstyle m_A} \\ B^6 & \xrightarrow{m_B\otimes 1\otimes 1\otimes 1\otimes\varepsilon_B} & B^4 & \xrightarrow{c\otimes c} & A^2 & \xrightarrow{m_A} & A \end{array} \tag{11.3.11.1}$$

The *multiplication induced by $c$* on $A\otimes B$ is the map $m_c : (A\otimes B)^2\to A\otimes B$ defined by linearity and

$$m_c(a\otimes b\otimes a'\otimes b') = \sum_{\bullet\bullet} aa'c(b_1\otimes b_1')\otimes b_2 b_2'$$

for all $a\otimes b\otimes a'\otimes b'\in(A\otimes B)^2$.

11.3.12. — Let $A$ be a commutative Hopf algebra over $K$ with unit $e : K \to A$, counit $\varepsilon\colon A \to K$, multiplication $m : A \otimes A \to A$, comultiplication $\mu : A \to A \otimes A$, and antipode $i : A \to A$. As is customary, we write the multiplication of two elements $m(a \otimes b)$ just as $ab$, and the comultiplication of an element $a$ as

$$\mu(a) = \sum_{\bullet} a_1 \otimes a_2 \qquad\qquad (1 \otimes \mu)(\mu(a)) = (\mu \otimes 1)(\mu(a)) = \sum_{\bullet} a_1 \otimes a_2 \otimes a_3$$

for as long as no confusion seems to arise (but maybe it's already too late for that concern). The category of representations of the affine group scheme $\operatorname{Spec} A$ is canonically equivalent to the category of $A$-comodules. As an abelian category, it only depends on $A$ as a coalgebra. The algebra structure on $A$ corresponds to the tensor product, and the existence of the antipode is equivalent to the existence of duals. We now seek to produce multiplications $m_\tau : A \otimes A \to A$ such that $(A, e, \varepsilon, \mu, m_\tau, i_\tau)$ is a commutative Hopf algebra for an antipode $i_\tau : A \to A$. Let us call *symmetric 2-cocycle* any symmetric bilinear map

$$\tau : A \otimes A \to K$$

which, seen as an element of the algebra $(A \otimes A)^\vee$, is invertible with inverse $\tau^{-1}$, and satisfies the following cocycle condition:

$$\sum_{\bullet\bullet} \tau(a_1 \otimes b_1)\tau(a_2 b_2 \otimes c) = \sum_{\bullet\bullet} \tau(a \otimes b_2 c_2)\tau(b_1 \otimes c_1)$$

As the nontion suggests, we can use such a cocycle in order to twist the originally given multiplication $m$ to a new multiplication $m_\tau$. It is defined by

$$m_\tau(a \otimes b) = \sum_{\bullet\bullet} \tau(a_1 \otimes b_1) a_2 b_2 \tau^{-1}(a_3 \otimes b_3)$$

for all $a, b \in A$, and we call it *twisted multiplication*. It will turn out that $A$, equipped with this twisted multiplication instead of the original one, is again a Hopf algebra. The new antipode will be given by

$$i_\tau(a) = \sum_{\bullet} \tau(a_1 \otimes i(a_2)) a_3 \tau^{-1}(i(a_4) \otimes a_5)$$

for $a \in A$, and we call it *twisted antipode*.

PROPOSITION 11.3.13. — *Let $A = (A, e, \varepsilon, \mu, m, i)$ be a commutative Hopf algebra and let $\tau : A \otimes A \to K$ be a symmetric 2-cocycle. With the twisted multiplication $m_\tau$, the twisted antipode $i_\tau$ and the original unit, counit and comultiplication, $A$ is a commutative Hopf algebra $(A, e, \varepsilon, \mu, m_\tau, i_\tau)$.*

PROOF. All required properties of $m_\tau$ and $i_\tau$ are straightforward to verify. Before we start checking a few of them, we notice that the map $\tau^{-1} : A \otimes A \to K$ is in general not a cocycle, but is symmetric and satisfies

$$\sum_{\bullet\bullet} \tau^{-1}(a_1 b_1 \otimes c)\tau^{-1}(a_2 \otimes b_2) = \sum_{\bullet\bullet} \tau^{-1}(a \otimes b_1 c_1)\tau^{-1}(b_2 \otimes c_2)$$

for all $a, b, c \in A$. With this relation in hand, we verify associativity of $m_\tau$.

$$\begin{aligned} m_\tau(m_\tau(a \otimes b) \otimes c) &= \sum_{\bullet\bullet} m_\tau(\tau(a_1 \otimes b_1)a_2b_2\tau^{-1}(a_3 \otimes b_3) \otimes c) \\ &= \sum_{\bullet\bullet\bullet} \tau(a_1 \otimes b_1)\tau(a_2b_2 \otimes c_2)a_3b_3c_3\tau^{-1}(a_4b_4 \otimes c_4)\tau^{-1}(a_5 \otimes b_5) \\ &= \sum_{\bullet\bullet\bullet} \tau(a_2 \otimes b_2c_2)\tau(b_1 \otimes c_1)a_3b_3c_3\tau^{-1}(a_4 \otimes b_4c_4)\tau^{-1}(b_5 \otimes c_5) \\ &= \sum_{\bullet\bullet} m_\tau(a \otimes \tau(b_1 \otimes c_1)b_2c_2\tau^{-1}(b_3 \otimes c_3)) \\ &= m_\tau(a \otimes m_\tau(b \otimes c)) \end{aligned}$$

That $m_\tau$ is commutative is an immediate consequence of the requirement that $\tau$ is symmetric. □

11.3.14. — Let $A = (A, e, \varepsilon, \mu, m, i)$ be a commutative Hopf algebra, and let $n : A \otimes A \to A$ be a symmetric bilinear map such that $A' = (A, e, \varepsilon, \mu, n, j)$ is a Hopf algebra, for some antipode $j$. Recall that if a bialgebra admits an antipode, it is unique. We want to fabricate a symmetric 2-cocycle $\tau$ such that $n = m_\tau$ holds.

Proposition 11.3.15. — *Let $H \xrightarrow{i} G \xrightarrow{p} F$ be morphisms of profinite groups.*

*(1) The morphism $i$ is injective if and only if for every finite $H$-set $S$ there exists a finite $G$-set $T$ and an injective map of $H$-sets $S \to T$.*
*(2) The morphism $p$ is surjective if and only if the functor $p^* : \mathbf{Set}(H) \to \mathbf{Set}(G)$ is full.*

Proof. □

Proposition 11.3.16. — *Let $G$ be a profinite group. Two closed subgroups $H$ and $N$ of $G$ are equal if and only if for every finite $G$-set $S$, the equality $S^H = S^N$ holds.*

Proof. If the closed subgroups $H$ and $N$ of $G$ are distinct, there exists an open normal subgroup $U$ of $G$ such that $H/(H \cap U)$ and $N/(N \cap U)$ are distinct in $G/U$. Up to replacing $G$ by $G/U$, we may thus assume without loss of generality that $G$ is finite. Let $S$ be the set of all subsets of $G$, on which $G$ acts by left translation: $gX = \{gx \mid x \in X\}$ for $X \in S$ a subset of $G$. The set $H \in S$ is a fixed point for the restricted action of $H$ on $X$, hence by assumption it is a fixed point for the action of $N$ on $G$. In other words, the equality $NH = H$ holds, whence $N \subseteq H$, and thus $N = H$ by symmetry. □

## 11.4. A fundamental exact sequence

We have introduced two canonical functors relating exponential Hodge structures to more benign objects. The first one is the inclusion $e\colon \mathbf{MHS} \to \mathbf{EMHS}$ sending an ordinary Hodge structure to corresponding constant exponential Hodge structure, and the second one is the functor $r\colon \mathbf{EMHS} \to \mathbf{Perv}_0$ associating with an exponential Hodge structure its underlying perverse sheaf. The functors

$$\mathbf{MHS} \xrightarrow{e} \mathbf{EMHS} \xrightarrow{r} \mathbf{Perv_0}$$

are compatible with tensor products and with fibre functors. The composite of these functors is the trivial functor. From the point of view of tannakian fundamental groups, this means that the two functors induce morphisms of group schemes $i\colon \pi_1(\mathbf{Perv}_0) \longrightarrow \pi_1(\mathbf{EMHS})$ and $p\colon \pi_1(\mathbf{EMHS}) \longrightarrow \pi_1(\mathbf{MHS})$ whose composite is the trivial morphism. The following theorem answers the question at hand.

Theorem 11.4.1. — *The sequence of group schemes over* $\mathbb{Q}$

$$\pi_1(\mathbf{Perv}_0) \xrightarrow{i} \pi_1(\mathbf{EMHS}) \xrightarrow{p} \pi_1(\mathbf{MHS}) \longrightarrow 1 \tag{11.4.1.1}$$

*induced by the canonical functors* $e\colon \mathbf{MHS} \to \mathbf{EMHS}$ *and* $r\colon \mathbf{EMHS} \to \mathbf{Perv}_0$ *is exact.*

11.4.2. — Before going into the proof, let us make two comments. First, the morphism $i$ : $\pi_1(\mathbf{Perv}_0) \longrightarrow \pi_1(\mathbf{EMHS})$ is not injective since there are objects in $\mathbf{Perv}_0$ not underlying a mixed Hodge module. However, if one starts with an object $M$ in **EMHS**, the fundamental group fits into an exact sequence

$$1 \longrightarrow \pi_1(\langle R_B(M)\rangle^{\otimes}) \longrightarrow \pi_1(\langle M\rangle^{\otimes}) \longrightarrow \pi_1(\langle M\rangle^{\otimes} \cap \mathbf{MHS}) \longrightarrow 1.$$

where $\langle - \rangle^{\otimes}$ stands for "tannakian category generated by". Indeed, we can understand the image of $i$ as the tannakian fundamental group of the tannakian subcategory of $\mathbf{Perv}_0$ generated by all objects which underly an exponential Hodge structure. Our second comment is that the surjective morphism $p$ has no section. Indeed, a section of $p$ would provide a functor of tannakian categories $\mathbf{EMHS} \to \mathbf{MHS}$ such that the composition with the canonical functor $c : \mathbf{MHS} \to \mathbf{EMHS}$ is isomorphic to the identity. But this is not possible, since in **EMHS** one has a square root of $\mathbb{Q}(-1)$ which does not exist in the category of mixed Hodge structures. However, as we will see in the next section, the corresponding exact sequence of Lie algebras is split.

11.4.3. — The proof of Theorem 11.4.1 relies on a general exactness criterion for fundamental groups of tannakian categories. One such criterion is given in [**EHS08**, Appendix] and another one in section 4 of [**dS12**]. The following proposition is a compromise between the two. Notice that condition (1) alone is not sufficient to ensure exactness. It is indeed equivalent to the statement that $\ker(p)$ is equal to the normal subgroup of $G$ generated by $\operatorname{im}(i)$, or also, that the GIT quotient $\ker(p)/\operatorname{im}(i)$ has no nonconstant regular functions. The typical example for this situation is the case where $H$ is a parabolic subgroup of $N = G$ and $F = \{1\}$.

Proof of Theorem 11.4.1. A morphism of affine group schemes $G \to F$ is surjective if and only if the corresponding functor $\mathbf{Rep}(F) \to \mathbf{Rep}(G)$ is fully faithful, with essential image stable under taking subobjects and quotients. Surjectivity of the morphism $p$ in the statement of the theorem follows thus from Lemma 11.2.4. It remains to show exactness in the middle. In order to apply the exactness criterion given in Proposition 11.3.2 we need to interpret categorically what invariants under the kernel of $p$ are. Let $G \to F$ be a surjective morphism of affine group schemes with kernel $N$. The functor $V \longmapsto V^N$ from representations of $G$ to representations of $F$ is right adjoint to the functor $\mathbf{Rep}(F) \to \mathbf{Rep}(G)$. Let thus $c : \mathbf{EMHS} \to \mathbf{MHS}$ be the right adjoint of the canonical functor $e$, and denote by $E(0)$ the unit object in $\mathbf{Perv}_0$. We need to verify that the two following statements are true.

(1) Let $M$ be an object of $\mathbf{EMHS}$. The morphism

$$\mathrm{Hom}_{\mathbf{Perv}_0}(E(0), rec(M)) \to \mathrm{Hom}_{\mathbf{Perv}_0}(E(0), r(M))$$

induced by the adjunction map $ec(M) \to M$ is an isomorphism.

(2) Let $E$ be a one dimensional object of $\mathbf{Perv}_0$ obtained as a subquotient of an object underlying an exponential Hodge structure. Then $E$ itself underlies an exponential Hodge structure.

Statement (1) follows formally from the existence of a six operations formalism for Hodge modules which is compatible with the six operations for perverse sheaves via the forgetful functors $r$ associating with a Hodge module on a variety its underlying perverse sheaf on the same variety. Let $\Pi\colon \mathbb{A}^1 \setminus \{0\} \to \operatorname{Spec} k$ be the structral morphism and $j\colon \mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1$ be the inclusion. The functor $e$ is the functor $j_!\pi^*$ from Hodge modules on the point to Hodge modules on the affine line. Its right adjoint is the functor $c = \pi_* j^!$. The functors $c$ and $e$ commut with $r$, so we find

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Perv}_0}(E(0), r(M)) &= \mathrm{Hom}_{\mathbf{Perv}_0}(e\mathbb{Q}, r(M)) \\
&= \mathrm{Hom}_{\mathbf{Vec}}(\mathbb{Q}, cr(M)) \\
&= \mathrm{Hom}_{\mathbf{Perv}_0}(e\mathbb{Q}, ecr(M)) \\
&= \mathrm{Hom}_{\mathbf{Perv}_0}(E(0), rec(M))
\end{aligned}$$

using that $E(0) = e\mathbb{Q}$ and that $e$ is fully faithful. As for statement (2), recall that a one dimensional object of $\mathbf{Perv}_0$ is determined up to isomorphism by the data of its only singularity $s \in k \subseteq \mathbb{C}$, and by the eigenvalue $\lambda \in \mathbb{Q}^\times$ of the local monodromy operator near $s$. The local monodromy operators of any Hodge module on $\mathbb{A}^1$ are quasiunipotent. Hence if a one dimensional object of $\mathbf{Perv}_0$ is a subquotient of an object underlying an exponential Hodge structure, then its local monodromy is either the identity, in which case it underlies the exponential Hodge structure $E(s)$, or its local monodromy is multiplication by $-1$, in which case it underlies the exponential Hodge structure $E(s) \otimes \mathbb{Q}(\frac{1}{2})$. □

## 11.5. The Hodge realisation of exponential motives

$$\mathrm{R}_{\mathrm{Hdg}} : \mathbf{M}^{\exp}(k) \longrightarrow \mathbf{EMHS} \tag{11.5.0.1}$$

CONJECTURE 11.5.1. — *The Hodge realisation functor* $\mathbf{M}^{\exp}(k) \longrightarrow \mathbf{EMHS}$ *is full.*

11.5.2. — Conjecture 11.5.1 enables us to control to a certain extent extension groups of exponential motives. For example, assuming the conjecture, the morphism of vector spaces

$$\mathrm{Ext}^1_{\mathbf{M}^{\exp}(k)}(M_1, M_2) \to \mathrm{Ext}^1_{\mathbf{EMHS}}(\mathrm{R}_{\mathrm{Hdg}}(M_1), \mathrm{R}_{\mathrm{Hdg}}(M_2))$$

is injective for all exponential motives $M_1$ and $M_2$. We can use this to gain some heuristics about the nature of the extension groups $\mathrm{Ext}^1_{\mathbf{M}^{\exp}(k)}(\mathbb{Q}(0), \mathbb{Q}(a))$ for integers $a$.

## 11.6. The vanishing cycles functor

Let $\mathbb{C}$ denote the complex affine line with coordinate $x$. For each $z \in \mathbb{C}$ and each mixed Hodge module $M$ on $\mathbb{A}^1$, the vanishing cycles $\varphi_{x-z}M$ form a mixed Hodge module on the point $\{z\}$, hence a mixed Hodge structure. We consider the functor:

$$\begin{aligned} \Phi\colon \mathbf{EMHS} &\longrightarrow \mathbf{MHS} \\ M &\longmapsto \bigoplus_{z\in\mathbb{C}} \varphi_{x-z}M. \end{aligned} \tag{11.6.0.1}$$

Observe that the sum is finite, since $\varphi_{x-z}M = 0$ unless $z$ is a singular point of $M$.

PROPOSITION 11.6.1. — *The functor* $\Phi$ *is compatible with the fibre functors.*

PROOF. □

The composition of $\Phi$ with the canonical functor $\mathbf{MHS} \to \mathbf{EMHS}$ is the identity. Observe that this refrains $\Phi$ from being a tensor functor, since $\mathbf{EMHS}$ contains a square root of the object $\Pi(i_*\mathbb{Q}(-1))$. To remedy this, we shall rather consider $\Phi$ with values in an enriched category, which takes into account the monodromy of vanishing cycles as well.

### 11.6.1. $\widehat{\mu}$-mixed Hodge structures.

DEFINITION 11.6.2. — A $\widehat{\mu}$-mixed Hodge structure is a pair $(H, T)$ consisting of a mixed Hodge structure and a finite order automorphism of mixed Hodge structures $T\colon H \to H$. Together with the obvious morphisms, $\mu$-mixed Hodge structures form a category which will be denoted by $\mathbf{MHS}^{\widehat{\mu}}$.

For each rational number $\alpha \in \mathbb{Q}$, let $H^\alpha = \ker(T - \exp(-2\pi i\alpha)) \subseteq H \otimes_{\mathbb{Q}} \mathbb{C}$, so there is a direct sum decomposition

$$H \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{\alpha\in\mathbb{Q}\cap(-1,0]} H^\alpha.$$

Following [**SS85**, p.661], we define the tensor product[1]

$$H_1 \otimes^{\mu} H_2$$

of two $\widehat{\mu}$-mixed Hodge structures $(H_1, T_1)$ and $(H_2, T_2)$ as follows:

(i) the underlying rational vector space is the tensor product of the underlying vector spaces $H_1 \otimes H_2$, together with the automorphism $T_1 \otimes T_2$;

(ii) the weight filtration is given by

$$W_k(H_1 \otimes^{\widehat{\mu}} H_2) = \Big(\bigoplus_{\alpha,\beta} \sum_{i,j} W_i H_1^{\alpha} \otimes W_j H_2^{\beta}\Big) \cap (H_1 \otimes H_2),$$

where the sum is over pairs of integers $(i, j)$ such that

$$i + j = \begin{cases} k & \text{if } \alpha = 0 \text{ or } \beta = 0, \\ k-2 & \text{if } \alpha + \beta = -1, \\ k-1 & \text{else;} \end{cases}$$

(iii) the Hodge filtration is given by

$$F^p(H_1 \otimes^{\widehat{\mu}} H_2) = \bigoplus_{\alpha,\beta} \sum_{k,\ell} F^k H_1^{\alpha} \otimes F^{\ell} H_2^{\beta},$$

where the sum is over pairs of integers $(k, \ell)$ such that

$$k + \ell = \begin{cases} p & \text{if } \alpha + \beta > -1, \\ p-1 & \text{if } \alpha + \beta \leqslant -1. \end{cases}$$

One checks that, equipped with these new filtrations, $H_1 \otimes^{\widehat{\mu}} H_2$ is again a mixed Hodge structure. Note that the inclusion $\mathbf{MHS} \to \mathbf{MHS}^{\widehat{\mu}}$ sending a Hodge structure $H$ to $(H, \mathrm{id})$ is a tensor functor, but the forgetful functor $\mathbf{MHS}^{\widehat{\mu}} \to \mathbf{MHS}$ is not.

**11.6.2. The enriched vanishing cycles functor.** Recall that each $\varphi_{x-z}M$ comes together with a monodromy operator $T$. If $T_s$ denotes its semisimple part, the pair $(\varphi_{x-z}M, T_s)$ defines a $\widehat{\mu}$-mixed Hodge structure. We get thus a functor with values in $\mathbf{MHS}^{\widehat{\mu}}$. The following theorem of Saito [**Sai**] asserts that it is compatible with the tensor structures on both sides:

THEOREM 11.6.3 (Saito). — *The functor* $\phi^{\widehat{\mu}} \colon \mathbf{EMHS} \to \mathbf{MHS}^{\widehat{\mu}}$ *is a tensor functor.*

REMARK 11.6.4. — Let $M$ be the square root of $\Pi(i_*\mathbb{Q}(-1))$ in **EMHS**. Then $\varphi^{\widehat{\mu}}(M)$ is the Hodge structure $\mathbb{Q}(0)$ equipped with the automorphism $-\mathrm{Id}$. Its tensor square is $\mathbb{Q}(-1)$ together with the trivial automorphism, which solves the problem we encountered before.

THEOREM 11.6.5. — *The corresponding exact sequence of Lie algebras is split, and a splitting is given by the vanishing cycles functor* $\mathbf{EMHS} \to \mathbf{MHS}$.

[1] This is called *join* in loc. cit. Note that there is a misprint in the definition of the weight filtration.

## 11.7. Monodromic exponential Hodge structures

Definition 11.7.1 (Kontsevich). — We call an exponential Hodge structure $M$ *monodromic* if $0 \in \mathbb{C}$ is its only singularity.

11.7.2. — A monodromic exponential Hodge structure is thus a Hodge module on the affine line whose fibre over $0 \in \mathbb{C}$ is trivial, and which is given by a variation of mixed Hodge structures on $\mathbb{C} \setminus \{0\}$. In other words, monodromic exponential Hodge structures are precisely those Hodge modules of the form $j_! V[1]$, where $j \colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$ is the inclusion and $V$ is a variation of mixed Hodge structures on $\mathbb{C} \setminus \{0\}$. The category of monodromic exponential Hodge structures is, as an abelian category, equivalent to the category of variations of Hodge structures on $\mathbb{C} \setminus \{0\}$.

## 11.8. The vanishing cycles functor

The vanishing cycles functor

$$\Psi : \mathbf{EMHS} \to \{\mathbb{C}\text{-graded monodromic exponential Hodge structures}\}$$

Theorem 11.8.1 (Kontsevich-Soibelmann). — *There exists a (non-canonical) natural isomorphism*

$$\Psi(M \otimes N) \cong \Psi(M) \otimes \Psi(N).$$

## 11.9. The weight filtration

Definition 11.9.1 (Kontsevich-Soibelman). — The *weight filtration* of an exponential mixed Hodge structure $M$ is defined by

$$W_n M = \Pi(W_n(M)),$$

where on the right hand side we regard $M$ as an object of $\mathbf{MHM}(\mathbb{C})$ and $W_n M$ denotes the weight filtration of mixed Hodge modules over $\mathbb{C}$.

Example 11.9.2. — It is instructive to examine the exponential Hodge structures $E(s)$ from Example 11.2.2. They are simple objects of **EMHS**, hence pure of some weight. However, regarded as objects of the bigger category $\mathbf{MHM}(\mathbb{C})$, the $E(s)$ are not simple, for they fit into an extension

$$0 \to \mathbb{Q}_s \to E(s) \to \mathbb{Q}[1] \to 0,$$

where $\mathbb{Q}_s$ denotes the skyscraper mixed Hodge module supported on $s$ with stalk $\mathbb{Q}(0)$. The above exact sequence describes the weight filtration of $E(s)$ as an object of $\mathbf{MHM}(\mathbb{C})$ as well:

$$W_0 E(s) = \mathbb{Q}_s \subseteq W_1 E(s) = E(s).$$

Since the graded piece $\mathrm{Gr}_1^W E(s) = \mathbb{Q}[1]$ is constant, it is killed by the projector $\Pi$ and one has $W_0E(s) = E(s)$ inside **EMHS**. We conclude that $E(s)$ is pure of weight 0.

11.9.3. — The weight filtration is functorial, and the functor $W_n\colon \mathbf{EMHS} \to \mathbf{EMHS}$ is exact, because the inclusion of **EMHS** into the category of mixed Hodge modules and the functor $\Pi$ are exact, and because the weight filtration is exact on mixed Hodge modules.

Proposition 11.9.4. — *The canonical functor $\iota : \mathbf{MHS} \to \mathbf{EMHS}$ is strictly compatible with the weight filtration. In other words, for every Hodge structure $H$ and integer $n$, the subobjects $W_n(\iota H)$ and $\iota(W_nH)$ of $\iota H$ are the same.*

Proof. Fix a Hodge structure $H$ and a weight $n$, and denote by $\alpha\colon \{0\} \to \mathbb{C}$ the inclusion and by $\pi\colon \mathbb{C} \to \{0\}$ the map to a point. Regarding $H$ as a Hodge module on the point $\{0\}$, there is a short exact sequence

$$0 \to \alpha_*H \to \iota H \to \pi^*H[1] \to 0$$

in $\mathbf{MHM}(\mathbb{C})$. In this sequence, the Hodge module $\pi^*H_{\mathbb{C}}[1]$ is the one defined by the constant variation of Hodge structures with fibre $H$, put in homological degree $-1$. We apply the exact functor $W_n$, and obtain the exact sequence

$$0 \to \alpha_*W_nH \to W_n\iota H \to \pi^*W_{n+1}H[1] \to 0$$

in $\mathbf{MHM}(\mathbb{C})$. Applying the exact functor $\Pi$ yields an isomorphism $\Pi(\alpha_*W_nH) \to \Pi(W_n\iota H)$. The functor $\iota$ is the composite $\Pi \circ \alpha_*$, so we find $\iota(W_nH) \xrightarrow{=} W_n(\iota H)$ where the weight filtration $W_n$ is now that in the category of exponential motives. □

Let $X$ be smooth. If the canonical map $H_c(X,f) \longrightarrow H(X,f)$ is an isomorphism, then the exponential mixed Hodge structure $H(X,f)$ is pure.

Example 11.9.5. —

(1) If $X$ is smooth and $f\colon X \longrightarrow \mathbb{A}^1$ is proper, then $H^n(X,f)$ is pure of weight $n$. .

(2) More generally, it suffices to assume that the function $f$ is *cohomologically tame* in the sense of Katz [**Kat90**, Prop. 14.13.3, item (2)]. This means that all the cohomology sheaves of the cone of the "forget supports" morphism $Rf_!\mathbb{Q}_X \to Rf_*\mathbb{Q}_X$ are lisse over $\mathbb{A}^1$ hence trivial. A more geometric condition, used by Sabbah [**Sab06**, §8], is that there exists an embedding $j\colon X \hookrightarrow Y$ into a smooth variety $Y$ and a proper map $\bar{f}\colon Y \to \mathbb{A}^1$ extending $f$ such that, for all $z \in \mathbb{C}$, the vanishing cycles complex $\varphi_{\bar{f}-z}(Rj_*\mathbb{Q}_X)$ is supported at a finite number of points lying in $X$ (as opposed to $Y$).

11.9.6 (The weight filtration is motivic). — Let $M$ be an exponential motive. The Hodge realisation of $M$, and hence its perverse realisation and its Betti realisation come equipped with a weight filtration. A natural question to ask is whether this filtration comes from a filtration of $M$ by submotives. If such a filtration exists, it is necessarily unique.

THEOREM 11.9.7. — *Every object $M$ of $\mathbf{M}^{\exp}(k)$ is equipped with an increasing and exhaustive filtration $W_\bullet M$ which maps to the weight filtration under the Hodge realisation functor.*

PROOF. We prove the theorem for motives $M$ of increasing generality. The cases we consider are, in summary, the following:

(1) $M = H^n(X, Y, f)$, where $f\colon X \to \mathbb{A}^1$ is proper.
(2) $M = H^n(X, Y, f)$ for arbitrary $X$, $Y$ and $f$
(3) $M$ an arbitrary exponential motive

Case 1: Let $X$ be a variety of dimension $\leqslant d$ with a proper morphism $f\colon X \to \mathbb{A}^1$, and let $Y \subseteq X$ be a subvariety of dimension $\leqslant d-1$. If $d = 0$, then $X$ is a collection of points and $Y$ is empty, and hence $H^n(X, Y, f)$ is pure of weight 0. Arguing by induction on dimension, we may suppose that the weight filtration on $H^{n-1}(Y)$ is motivic, with weights $0, 1, \ldots, n-1$. By resolution of singularities, there is a smooth variety $\widetilde{X}$ of dimension $d$ mapping to $X$ with a normal crossings divisor $\widetilde{Y}$ mapping to $Y$ such that

$$H^n(X, Y, f) \to H^n(\widetilde{X}, \widetilde{Y}, \widetilde{f})$$

is an isomorphism. We may thus suppose without loss of generality that $X$ is smooth and $Y$ a normal crossings divisor. From the long exact sequence

$$\cdots \longrightarrow H^{n-1}(Y, f|_Y) \longrightarrow H^n(X, Y, f) \longrightarrow H^n(X, f) \longrightarrow \cdots$$

and the fact that $H^n(X, f)$ is pure of weight $n$, we see that the weight filtration on $H^n(X, Y, f)$ is given by

$$\begin{aligned} W_s H^n(X, Y, f) &= \operatorname{im}(W_s H^{n-1}(Y, f|_Y) \to H^n(X, Y, f)) && \text{for } s < n \\ W_s H^n(X, Y, f) &= H^n(X, Y, f) && \text{for } s \geqslant n \end{aligned}$$

hence is motivic. In particular, the weights of $H^n(X, Y, f)$ are $0, 1, \ldots, n$.

Case 2: We now treat the case of a motive $M$ of the form $M = H^n(X, Y, f)$ for a smooth, not necessarily proper variety $X$ with a function $f\colon X \to \mathbb{A}^1$, and a smooth subvariety $Y \subseteq X$. We choose a smooth relative compactification $\overline{f}\colon \overline{X} \to \mathbb{A}^1$. That means the following: $X$ is an open subvariety of a smooth variety $\overline{X}$ with complement a normal crossings divisor $D$, the closure $\overline{Y}$ of $Y$ in $\overline{X}$ is smooth and $D + \overline{Y}$ has normal crossings, and $\overline{f}\colon \overline{X} \to \mathbb{A}^1$ is a proper map extending $f$. Let $D_1, D_2, \ldots, D_N$ be the smooth components of the divisor $D$, and set

$$X^{(p)} = \bigsqcup_{1 \leqslant i_1 < \cdots < i_p \leqslant N} D_{i_1} \cap \cdots \cap D_{i_p}$$

for $p = 0, 1, \ldots, N$, and $Y^{(p)} = X^{(p)} \cap Y$. In particular we set $X^{(0)} = \overline{X}$. The varieties $X^{(p)}$ and $Y^{(p)}$ are smooth, and there are inclusions maps $\iota_s\colon (X^{(p)}, Y^{(p)}) \to (X^{(p-1)}, Y^{(p)})$ for $s = 1, 2, \ldots, p$. We use alternating sums of the induced Gysin morphisms (4.7.3.3) to get a double complex

$$\cdots \to C^*(X^{(p)}, Y^{(p)}, f)[2p](p) \to C^*(X^{(p-1)}, Y^{(p-1)}, f)[2p-2](p-1) \to \cdots \to C^*(\overline{X}, \overline{Y}, \overline{f})$$

The total complex of this double complex computes the cohomology of $(X, Y, f)$. This is where the spectral sequence

$$E_2^{p,q} = H^{2p+q}(X^{(p)}, Y^{(p)}, f)(p) \implies H^{p+q}(X, Y, f)$$

comes from.

Case 3:

□

## 11.10. The irregular Hodge filtration

In this paragraph, we recall that the de Rham cohomology $H^*_{\mathrm{dR}}(X, f)$ is equipped with an *irregular Hodge filtration* which is indexed by rational numbers and has finitely many jumps. It was first introduced by Deligne [**Del07**] in the case of curves, then generalized to higher dimensional varieties by Yu [**Yu14**]. Further properties—especially the degeneration of the corresponding spectral sequence—were studied by Sabbah, Esnault and Yu in [**Sab10**] and [**ESY17**].

11.10.1 (The Kontsevich complex). — Let $X$ be a smooth variety of dimension $n$ over $k$, together with a regular function $f$, and let $\overline{X}$ be a good compactification of $(X, f)$ as in Definition 3.5.8. We keep the same notation from *loc. cit.*, so $D = \overline{X} \setminus X$ is the normal crossing divisor at infinity and $P$ the pole divisor of $f$. We write $P = \sum e_i P_i$ with $P_i$ the irreducible components. The connection $\mathcal{E}^f$ on $X$ extends to an integrable meromorphic connection on $\overline{X}$ with associated de Rham complex

$$(\Omega^\bullet_{\overline{X}}(*D), d_f).$$

However, the subsheaves $\Omega^p_{\overline{X}}(\log D) \subseteq \Omega^p_{\overline{X}}(*D)$ of logarithmic differentials do *not* form a subcomplex, so one cannot naively imitate the constructions from Hodge theory.

A possible way to circumvent this problem, after Kontsevich, is as follows: given a rational number $\alpha \in [0, 1) \cap \mathbb{Q}$, we set $[\alpha P] = \sum [\alpha e_i] P_i$, where $[\cdot]$ stands for the integral part, and

$$\Omega^p_{\overline{X}}(\log D)([\alpha P]) = \Omega^p_{\overline{X}}(\log D) \otimes_{\mathcal{O}_{\overline{X}}} \mathcal{O}_{\overline{X}}([\alpha P]). \tag{11.10.1.1}$$

We then define a subsheaf $\Omega^p_f(\alpha)$ of (11.10.1.1) by asking that, for every open subset $U \subseteq \overline{X}$,

$$\Omega^p_f(\alpha)(U) = \{\omega \in \Omega^p_{\overline{X}}(\log D)([\alpha P])(U) \mid df \wedge \omega \in \Omega^{p+1}_{\overline{X}}(\log D)([\alpha P])(U)\}.$$

In particular, one has:

$$\Omega^0_f(\alpha) = \mathcal{O}_{\overline{X}}([\alpha - 1]P), \quad \Omega^n_f(\alpha) = \Omega^n_{\overline{X}}(\log D)([\alpha P]).$$

The sheaves $\Omega^p_f(\alpha)$ are now obviously stable under $d_f$ and form a complex which computes the de Rham cohomology of the pair $(X, f)$:

Proposition 11.10.2. — *The inclusion $(\Omega^\bullet_f(\alpha), d_f) \hookrightarrow (\Omega^\bullet_{\overline{X}}(*D), d_f)$ is a quasi-isomorphism for each $\alpha \in [0, 1) \cap \mathbb{Q}$. In particular, there are canonical isomorphisms*

$$H^n_{\mathrm{dR}}(X, f) \cong H^n(\overline{X}, (\Omega^\bullet_f(\alpha), d_f)). \tag{11.10.2.1}$$

11.10.3. —

Definition 11.10.4. — The *irregular Hodge filtration* is given by

$$F^{p-\alpha}H^n_{\mathrm{dR}}(X,f) = \mathrm{Im}(\mathbb{H}^n(\overline{X},(\Omega_f^{\bullet\geqslant p}(\alpha),d_f)) \longrightarrow \mathbb{H}^n(\overline{X},(\Omega_f^{\bullet}(\alpha),d_f))). \tag{11.10.4.1}$$

In fact, the relevant $\alpha$ will be those of the form $\alpha = \frac{\ell}{m}$ where $m$ is the multiplicity of an irreducible component of $P$ and $\ell = 1,\dots,m-1$.

11.10.5. — Let us compute a few examples of irregular Hodge filtrations:

11.10.6 (Compatibility with the Künneth formula). — We now assume that we are given two pairs $(X_1,f_1)$ and $(X_2,f_2)$ consisting of smooth varieties over $k$ and regular functions. As usual, we consider the cartesian product $X_1\times X_2$ together with the Thom–Sebastiani sum $f_1\boxplus f_2$. By the Künneth formula, cup-product induces an isomorphism of $k$-vector spaces

$$\bigoplus_{i+j=n} H^i_{\mathrm{dR}}(X_1,f_1)\otimes H^j_{\mathrm{dR}}(X_2,f_2) \longrightarrow H^n_{\mathrm{dR}}(X_1\times X_2, f_1\boxplus f_2). \tag{11.10.6.1}$$

We equip the left-hand side of (11.10.6.1) with the product filtration, that is

$$\bigoplus_{i+j=n}\Bigl(\sum_{a+b=\lambda} F^a H^i_{\mathrm{dR}}(X_1,f_1)\otimes F^b_{\mathrm{dR}}(X_2,f_2)\Bigr) \tag{11.10.6.2}$$

Theorem 11.10.7 (Chen-Yu, [**CY**]). — *The map* (11.10.6.1) *is an isomorphism of filtered vector spaces.*

CHAPTER 12

# Examples and consequences of the period conjecture

In this chapter, we present a number of explicit computations of Galois groups of exponential motives which provide some evidence supporting the period conjecture 8.2.3.

## 12.1. Exponentials of algebraic numbers

Arguably, the most elementary exponential period which is not – or at least seems not to be – a period in the classical sense is the base of the natural logarithm $e$. That $e$ is an irrational number was known to Euler, and its transcendence was shown by Hermite in 1873. The Lindemann-Weierstrass theorem, which we recall below, generalises Hermite's transcendence theorem. We will show that the Lindemann-Weierstrass theorem is a consequence of the exponential period conjecture, hence serves as an illustration of it. We will also show that the period conjecture implies that $e$ is not a period in the classical sense, and in fact even algebraically independent from classical periods.

Theorem 12.1.1 (Lindemann-Weierstrass). — *Let $\alpha_1, \ldots, \alpha_n$ be algebraic numbers and consider the $\mathbb{Q}$-vector space $\langle \alpha_1, \ldots, \alpha_n \rangle_{\mathbb{Q}} \subseteq \overline{\mathbb{Q}}$ generated by them. Then:*

$$\operatorname{trdeg} \overline{\mathbb{Q}}(e^{\alpha_1}, \ldots, e^{\alpha_n}) = \dim_{\mathbb{Q}} \langle \alpha_1, \ldots, \alpha_n \rangle.$$

*In particular, if $\alpha_1, \ldots, \alpha_n$ are $\mathbb{Q}$-linearly independent, then their exponentials $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are algebraically independent.*

12.1.2. — Let us now explain how one can see the Lindemann-Weierstrass theorem as an instance of the exponential period conjecture. Given algebraic numbers $\alpha_1, \ldots, \alpha_n$, consider

$$E(\alpha_1, \ldots, \alpha_n) = E(\alpha_1) \oplus \cdots \oplus E(\alpha_n),$$

where $E(\alpha_i)$ denotes the one-dimensional exponential motive over $\overline{\mathbb{Q}}$ defined by

$$E(\alpha_i) = H^0(\operatorname{Spec}(\overline{\mathbb{Q}}), -\alpha_i).$$

In particular $E(0) = \mathbb{Q}(0)$ is the unit motive. The period algebra of this motive is generated by the exponentials $e^{\alpha_1}, \ldots, e^{\alpha_n}$, and the period conjecture predicts that the transcendence degree over $\overline{\mathbb{Q}}$ of this algebra is equal to the dimension of the tannakian fundamental group of the motive $E(\alpha_1, \ldots, \alpha_n)$.

PROPOSITION 12.1.3. — *Let $\alpha_1, \ldots, \alpha_n$ be algebraic numbers. The Galois group of the exponential motive $E(\alpha_1, \ldots, \alpha_n)$ is a split torus of dimension* $\dim_{\mathbb{Q}}\langle\alpha_1, \ldots, \alpha_n\rangle_{\mathbb{Q}}$.

PROOF. For every $\alpha \in \overline{\mathbb{Q}}$, the motive $E(\alpha)$ is one dimensional. Its tannakian fundamental group is thus canonically isomorphic to a subgroup of $\mathbb{G}_m$. This allows us to canonically identify the fundamental group $G$ of $E(\alpha_1, \ldots, \alpha_n)$ with a subgroup of $\mathbb{G}_m^n$. We will show that $G \subseteq \mathbb{G}_m^n$ is equal to the subtorus $T \subseteq \mathbb{G}_m^n$ whose group of characters is the subgroup of $\overline{\mathbb{Q}}$ generated by $\alpha_1, \ldots, \alpha_n$, which we view as a quotient of $\mathbb{Z}^n$.

There is a canonical isomorphism of motives $E(\alpha) \otimes E(\beta) \cong E(\alpha+\beta)$ for all algebraic numbers $\alpha$, $\beta$. By induction, every $\mathbb{Z}$-linear relation $c_1\alpha_1 + \cdots + c_n\alpha_n = 0$ yields an isomorphism of motives:

$$E(\alpha_1)^{\otimes c_1} \otimes \cdots \otimes E(\alpha_n)^{\otimes c_n} \cong E(0) = \mathbb{Q}(0)$$

The action of the Galois group on the right-hand side is trivial, hence it must be trivial on the left-hand side as well. Thus, if $(z_1, \ldots, z_n) \in \mathbb{G}_m^n$ lies in $G$, then $z_1^{c_1} z_2^{c_2} \cdots z_n^{c_n} = 1$. This yields the inclusion $G \subseteq T$.

In order to establish the inclusion $T \subseteq G$, recall the Galois group of a motive $M$ contains the Galois group of its perverse realisation. Set $F(\alpha) := \mathrm{R}_{\mathrm{perv}}(E(\alpha))$, and let us show that the Galois group of $F(\alpha_1, \ldots, \alpha_n) = F(\alpha_1) \oplus \cdots \oplus F(\alpha_n)$ in $\mathbf{Perv}_0$ is already all of $T$. All objects in the tannakian category generated by $F$ are semisimple, and simple objects are precisely those one-dimensional objects of the form

$$F(\alpha) = F(\alpha_1)^{\otimes c_1} \otimes \cdots \otimes F(\alpha_n)^{\otimes c_n}$$

where $\alpha = c_1\alpha_1 + \cdots + c_n\alpha_n$ is a linear combination of the algebraic numbers $\alpha_1, \ldots, \alpha_n$. The claim now follows from the fact that for any two complex numbers $\alpha$ and $\beta$ we have $\mathrm{Hom}(F(\alpha), F(\beta)) = 0$ unless $\alpha = \beta$. In other words, the tannakian category generated by $F(\alpha_1, \ldots, \alpha_n)$ is equivalent to the category of rational vector spaces with a grading indexed by the finitely generated group $\langle\alpha_1, \ldots, \alpha_n\rangle_{\mathbb{Z}}$. □

PROPOSITION 12.1.4. — *Assume that the exponential period conjecture 8.2.3 holds. Then the exponential of a non-zero algebraic number is transcendental over the field of usual periods.*

PROOF. Let $\mathbb{P}$ be the field generated by the periods of usual motives. We need to show that, given a non-zero algebraic number $\alpha$ and a polynomial $f \in \mathbb{P}[x]$, the relation $f(e^\alpha) = 0$ implies $f = 0$. We choose a usual motive $M$ over $\overline{\mathbb{Q}}$ such that all the coefficients of $f$ lie in the field generated by the periods of $M$ and we consider the exponential motive $M^+ = M \oplus E(\alpha)$. Its field of periods is generated by all the periods of $M$ together with $e^\alpha$. Let $G_M$ and $G_{M^+}$ denote the corresponding motivic Galois groups. Since $\langle M\rangle^{\otimes}$ is a subcategory of $\langle M^+\rangle^{\otimes}$, there is a canonical surjection $G_{M^+} \to G_M$. Assuming the exponential period conjecture, it suffices to prove that the inequality $\dim G_{M^+} > \dim G_M$ holds.

Let $F$ and $F^+$ be the perverse realisations of $M$ and $M^+$, and denote by $G_{F^+}$ and $G_F$ their Galois groups. The group $G_F$ is trivial, since $F$ comes from a usual motive, hence is isomorphic to a sum of copies of the neutral object in the tannakian category $\mathbf{Perv}_0$. The group $G_{F^+}$ is the same as $G_{E(\alpha)}$, hence is isomorphic to $\mathbb{G}_m$ since $\alpha \neq 0$. The diagram

$$\begin{array}{ccc} \mathbb{G}_m \cong G_{F^+} & \twoheadrightarrow & G_F = 0 \\ \downarrow \subseteq & & \downarrow \subseteq \\ G_{M^+} & \twoheadrightarrow & G_M \end{array}$$

shows that the surjection $G_{M^+} \to G_M$ contains a copy of $\mathbb{G}_m$ in its kernel, hence the sought inequality of dimensions. □

12.1.5. — The exponential motives $E(\alpha)$ are sufficiently easy for us to understand their reductions modulo primes and to write down their $L$-function. Understanding the $L$-function is another matter. To simplify, let us assume $\alpha$ is a rational integer, so that $E(\alpha)$ is a motive over the number field $k = \mathbb{Q}$.

$$a_p = \exp\left(\tfrac{2\pi i}{p}\right)$$

$$a_n = \exp\Big(2\pi i \sum_p v_p(n) p^{-1}\Big)$$

$$L_E = \prod_p \frac{1}{1 - a_p p^{-s}} = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

Proposition 12.1.6. — *The function $L_E(s)$ is meromorphic on the half-plane $\mathrm{Re}(s) > 0$ except at $s = 1$ where $L_E(s)$ has a pole of order $1$. The zeroes of $L_E(s)$ in the half-plane $\mathrm{Re}(s) > 0$ are the same as those of the Riemann zeta function.*

Proof. It suffices to show that the quotient $L_E(s)/\zeta(s)$ is holomorphic and non-zero on the half-plane $\mathrm{Re}(s) > 0$. This quotient can be written as an Euler product

$$\prod_p \frac{1 - p^{-s}}{1 - a_p p^{-s}} = \prod_p 1 - \frac{p^{-s}(1 - a_p)}{1 - a_p p^{-s}} \tag{12.1.6.1}$$

which converges absolutely for $\mathrm{Re}(s) > 0$. The convergence follows from the elementary estimates $|1 - a_p| < 2\pi p^{-1}$ and $|1 - a_p p^{-s}| \geqslant 1 - p^{-\mathrm{Re}(s)} \geqslant 1 - 2^{-\mathrm{Re}(s)}$ which yield

$$\left|\frac{p^{-s}(1 - a_p)}{1 - a_p p^{-s}}\right| \leqslant \frac{p^{-\mathrm{Re}(s)}\, 2\pi p^{-1}}{1 - 2^{-\mathrm{Re}(s)}} \leqslant C p^{-\mathrm{Re}(s) - 1} \tag{12.1.6.2}$$

for a constant $C = C(s)$ depending on $s$ but not on $p$. From this estimate we deduce that the sum over $p$ of the terms (12.1.6.2) converges, hence so does the Euler product (12.1.6.1). □

## 12.2. The motive $\mathbb{Q}(-\frac{1}{2})$

There has already been some speculation about motives $\mathbb{Q}(-\frac{1}{2})$ and $\mathbb{Q}(-\frac{1}{4})$ over finite fields[1]. Specifically, if $E$ is a supersingular elliptic curve over a finite field and $F$ a field of coefficients splitting the quaternion algebra $\mathrm{End}(E)$, then the 2-dimensional motive $H^1(E)$ decomposes as a sum $M \oplus M$ where $M \otimes M$ is isomorphic to the Tate motive $F(-1) = H^2(\mathbb{P}^1)$.

Over a field of characteristic zero, motives $M$ with $M \otimes M \simeq \mathbb{Q}(-1)$ should not exist, at least not in the classical sense, since the Hodge realisation of such an $M$ would necessarily be a one-dimensional Hodge structure of weight 1. This is why it is not expected that $\sqrt{2\pi i}$ is a period in the classical sense. However, we can easily write $\sqrt{2\pi i}$ and $\sqrt{\pi}$ as periods of exponential motives over $\mathbb{Q}(i)$ and over $\mathbb{Q}$ respectively:

$$\sqrt{2\pi i} = \int_{(i+1)\mathbb{R}} e^{\frac{-1}{2i}x^2}\,dx,$$
$$\sqrt{\pi} = \Gamma(\tfrac{1}{2}) = \int_{\mathbb{R}} e^{-x^2}\,dx.$$

The corresponding exponential motives are $H^1(\mathbb{A}^1_{\mathbb{Q}(i)}, \frac{1}{2i}x^2)$ and $H^1(\mathbb{A}^1_{\mathbb{Q}}, x^2)$ respectively, where $x$ is the coordinate of the affine line. This suggests that the motive $H^1(\mathbb{A}^1_k, \frac{1}{2i}x^2)$ is a reasonable candidate for $\mathbb{Q}(-\frac{1}{2})$. We will show that indeed for any field $k \subseteq \mathbb{C}$ and non-zero element $a \in k$ such that $a = \frac{1}{2i}c^2$ for some $c \in k$, there is an isomorphism

$$H^1(\mathbb{A}^1_k, ax^2) \otimes H^1(\mathbb{A}^1_k, ax^2) \cong \mathbb{Q}(-1)$$

of exponential motives over $k$. Given any two non-zero elements $a$ and $b$ of $k$, the motives $H^1(\mathbb{A}^1_k, ax^2)$ and $H^1(\mathbb{A}^1_k, bx^2)$ are isomorphic if and only if $a = c^2 b$ for some $c \in k^*$. It follows that if $k$ contains $i$, there exist motives over $k$ whose tensor square is $\mathbb{Q}(-1)$, and indeed many of them unless $k$ is quadratically closed. Let us fix

$$M(\sqrt{\pi}) := H^1(\mathbb{A}^1, x^2)$$

as a particular exponential motive over $k$ with period $\sqrt{\pi}$.

LEMMA 12.2.1. — *Let $a, b \in k^\times$ and let $C_{a,b}$ be the affine conic over $k$ defined by the equation $as^2 + bt^2 = 1$. There exists an isomorphism of exponential motives*

$$H^1(\mathbb{A}^1, ax^2) \otimes H^1(\mathbb{A}^1, bx^2) \simeq H^1(C_{a,b}).$$

PROOF. By the Künneth formula, it suffices to show that $H^2(\mathbb{A}^2, ax^2 + by^2)$ and $H^1(C_{a,b})$ are isomorphic. At the level of periods, this is reflected by the identity

$$\int_{e^{i\arg(a)}\mathbb{R}\times e^{i\arg(b)}\mathbb{R}} e^{-ax^2-by^2}\,dxdy = \frac{\pi}{\sqrt{ab}},$$

which follows from the change of coordinates $x = r\cos\theta$ and $y = r\sin\theta$. Inspired by this, we consider the morphism $h\colon C_{a,b} \times \mathbb{A}^1 \to \mathbb{A}^2$ given by $h((s,t), r) = (rs, rt)$. Since $h$ sends the

[1]For $\mathbb{Q}(1/2)$, see [**Ram05**] and references given there (Milne). For $\mathbb{Q}(1/4)$, see [**Jos10**]

subvariety $C_{a,b}\times\{0\}$ to $\{(0,0)\}$ and commutes with the functions $0\boxplus r^2$ on the source and $ax^2+by^2$ on the target, it induces a morphism of exponential motives

$$h\colon H^2(\mathbb{A}^2,\{(0,0)\},ax^2+by^2)\longrightarrow H^2(C_{a,b}\times\mathbb{A}^1,C_{a,b}\times\{0\},0\boxplus r^2).$$

Noting that the left hand side is isomorphic to $H^2(\mathbb{A}^2,ax^2+by^2)$ by the exact sequence (4.2.4.2) associated to the immersions $\emptyset\subseteq\{(0,0)\}\subseteq\mathbb{A}^2$ and applying the Künneth formula again, we get

$$h'\colon H^2(\mathbb{A}^2,ax^2+by^2)\longrightarrow H^1(C_{a,b})\otimes H^1(\mathbb{A}^1,\{0\},r^2).$$

Now the last factor fits into an exact sequence of motives

$$0\to\mathbb{Q}(0)\to H^1(\mathbb{A}^1,\{0\},r^2)\to H^1(\mathbb{A}^1,r^2)\to 0.$$

We will show that the second component of $h'$ vanishes in $H^1(\mathbb{A}^1,r^2)$, and that the induced map $H^2(\mathbb{A}^2,ax^2+by^2)\to H^1(C_{a,b})$ is an isomorphism. For this it suffices to work in a realisation: for instance, de Rham cohomology. There $h'$ sends the generator $dxdy$ to $(tds-sdt)\otimes rdr$. The first factor is a generator of $H^1_{\mathrm{dR}}(C_{a,b})$ and the second vanishes in $H^1_{\mathrm{dR}}(\mathbb{A}^1,r^2)$ since it is equal to $\frac{1}{2}d_{r^2}(1)$. However, it is non-zero in $H^1_{\mathrm{dR}}(\mathbb{A}^1,\{0\},r^2)$, as one can see from the integral $\int_0^{+\infty}e^{-r^2}rdr=1$. □

In particular, $M(\sqrt{\pi})^{\otimes 2}=H^1(s^2+t^2=1)$.

12.2.2. — If the base field $k$ contains a square root of $-1$, the conic $C$ is isomorphic to $\mathbb{G}_m$ by the change of coordinates $u=s+it, v=s-it$, and therefore $M(\sqrt{\pi})$ is a genuine tensor square root of $\mathbb{Q}(-1)$. We can generalise Lemma 12.2.1 to the case where in place of $M(\sqrt{\pi})$ we consider a motive of the form $H^n(\mathbb{A}^n,q)$ for a quadratic form $q$ in $n$ variables $x_1,\dots,x_n$, seen as a regular function on $\mathbb{A}^n=\operatorname{Spec}k[x_1,\dots,x_n]$. Given a non-zero element $c\in k$, we define

$$M(\sqrt{c})=\begin{cases}H^0(\operatorname{Spec}k(\sqrt{c}))/\mathbb{Q}(0) & \text{if } c \text{ is not a square in } k,\\ \mathbb{Q}(0) & \text{if } c \text{ is a square in } k.\end{cases}\tag{12.2.2.1}$$

The motive $M(\sqrt{c})$ is one dimensional, and only depends on the class of $c$ modulo squares.

Proposition 12.2.3. — *Let $q=q(x_1,\dots,x_n)$ be a non-degenerate quadratic form, seen as a regular function on $\mathbb{A}^n=\operatorname{Spec}k[x_1,\dots,x_n]$. Then $H^m(\mathbb{A}^n,q)=0$ for $m\neq n$ and*

$$H^n(\mathbb{A}^n,q)\cong M(\sqrt{\det q})\otimes M(\sqrt{\pi})^{\otimes n}.$$

Proof. It is a standard fact that there exists a linear automorphism of $\mathbb{A}^n$ transforming any given quadratic form into a diagonal one. Thus, we may assume that $q$ is of the form

$$q(x_1,\dots,x_n)=a_1x_1^2+a_2x_2^2+\cdots+a_nx_n^2$$

for some non-zero elements $a_1,\dots,a_n\in k$. The discriminant of $q$ is the product $a_1a_2\cdots a_n$, which does not depend on the diagonalization modulo $(k^\times)^2$. The Künneth formula yields

$$H^n(\mathbb{A}^n,q)\cong H^1(\mathbb{A}^1,a_1x^2)\otimes H^1(\mathbb{A}^1,a_2x^2)\otimes\cdots\otimes H^1(\mathbb{A}^1,a_nx^2)$$

and $H^m(\mathbb{A}^n,q)=0$ for $m\neq n$. The result then follows from $H^1(\mathbb{A}^1,a_ix^2)\cong M(\sqrt{a_i})\otimes M(\sqrt{\pi})$. □

12.2.4 (The $\ell$-adic realisation). — Let $\chi_2\colon \mathbb{F}_q^\times \to \{\pm 1\}$ be the non-trivial quadratic character on $\mathbb{F}_q^\times$. Given an additive character $\psi$, one defines the *Gauss sum*

$$G(\chi_2,\psi) = \sum_{x\in\mathbb{F}_q^\times} \chi_2(x)\psi(x). \tag{12.2.4.1}$$

Lemma 12.2.5. — *The exponential motive $H^1(\mathbb{A}^1, x^2)$ has good ramification outside $p=2$ and its $\ell$-adic realisation is the one-dimensional $\overline{\mathbb{Q}}_\ell$-vector space with Frobenius action given by multiplication by $G(\chi_2,\psi)$.*

Proof. The $\ell$-adic perverse realisation of $H^1(\mathbb{A}^1, x^2)$ is $j_!\mathcal{L}_{\chi_2}[1]$. □

The $L$-function of $M(\sqrt{\pi})$ is given by the Euler-product

$$L(s) = \prod_p \frac{1}{1-g_p p^{-s+1/2}} \qquad \text{with} \qquad g_p = \begin{cases} 0 & \text{if } p=2 \\ 1 & \text{if } p\equiv 1 \bmod 4 \\ i & \text{if } p \equiv 3 \bmod 4 \end{cases}$$

Extending the $g_p$ to a multiplicative function on integers, we may write $L(s)$ as a Dirichlet series.

$$L(s) = \sum_{n\geqslant 1 \text{ odd}} \frac{\exp(2\pi i\frac{r(n)}{4})}{n^{s-1/2}}$$

where $r(n)$ stands for

$$\sum_{p\equiv 3\ (4)} v_p(n)$$

The coefficient $\exp(2\pi i\frac{r(n)}{4})$ depends only on the class of $r(n)$ modulo 4. The Dirichlet series converges absolutely for $\mathrm{Re}(s) > \frac{3}{2}$, and does not converge absolutely for $s=\frac{3}{2}$. For $r=0,1,2,3$ and $x>0$, define the following sets of odd integers:

$$A(r,x) = \{n<x \text{ odd} \mid r(n)\equiv r \bmod 4\}$$

Conjecture 12.2.6. — *For $r=0,1,2,3$, the limit*

$$A(r) := \lim_{x\to\infty} \frac{2\#A(r,x)}{x}$$

*exists, and we have $A(0)+A(2)=\frac{1}{2}$ and $A(1)+A(3)=\frac{1}{2}$. Moreover $A(0)<\frac{1}{4}$ and $A(1)>\frac{1}{4}$ holds.*

We counted the number of odd integers $n$ between $N$ and $N + 10^8$ for which $r(n)$ is congruent to $0, 1, 2, 3$. The numerical calculation suggests the following approximate values

| | $4A(0)$ | $4A(1)$ | $4A(2)$ | $4A(3)$ |
|---|---|---|---|---|
| $N = 10^9$ | 0.8201 | 1.3632 | 1.1799 | 0.6368 |
| $N = 10^{10}$ | 0.8114 | 1.3375 | 1.1886 | 0.6625 |
| $N = 10^{11}$ | 0.8048 | 1.3143 | 1.1952 | 0.6857 |
| $N = 10^{12}$ | 0.8000 | 1.2937 | 1.2000 | 0.7063 |
| $N = 10^{13}$ | 0.7964 | 1.2751 | 1.2036 | 0.7249 |
| $N = 10^{14}$ | 0.7939 | 1.2584 | 1.2061 | 0.7416 |
| $N = 10^{15}$ | 0.7921 | 1.2434 | 1.2079 | 0.7566 |
| $N = 10^{16}$ | 0.7911 | 1.2294 | 1.2089 | 0.7706 |
| $N = 10^{17}$ | 0.7906 | 1.2168 | 1.2094 | 0.7832 |

## 12.3. Exponential periods on the affine line

Set $\mathbb{A}^1 = \operatorname{Spec} k[x]$ and let $f \in k[x]$ be a polynomial of degree at least two. In this section, we study the motive $H^1(\mathbb{A}^1, f)$ and its motivic Galois group. In particular, we want to understand the determinant of $H^1(\mathbb{A}^1, f)$.

12.3.1. — In tannakian terms, exterior powers are constructed as follows. For any object $M$ of a tannakian category and any integer $n \geqslant 1$, the symmetric group $\mathfrak{S}_n$ acts on the $n$-fold tensor power $M^{\otimes n}$ by permutation of factors. The $n$-fold exterior power of $M$ is the eigenspace in $M^{\otimes n}$ of the signature character $\varepsilon\colon \mathfrak{S}_n \to \{\pm 1\}$. Given a non-constant polynomial $f$, the $n$-fold tensor power of the exponential motive $H^1(\mathbb{A}^1, f)$ can be identified with $H^n(\mathbb{A}^n, f^{\boxplus n})$ via the Künneth isomorphism

$$\kappa\colon H^1(\mathbb{A}^1, f)^{\otimes n} \xrightarrow{\cong} H^n(\mathbb{A}^n, f^{\boxplus n}) \tag{12.3.1.1}$$

because $H^q(\mathbb{A}^1, f) = 0$ for $q \neq 1$. The symmetric group $\mathfrak{S}_n$ acts on $\mathbb{A}^n$ by permutation of coordinates, and this action commutes with the Thom–Sebastiani sum $f^{\boxplus n} = f \boxplus \cdots \boxplus f$, hence an action of $\mathfrak{S}_n$ on the motive $H^n(\mathbb{A}^n, f^{\boxplus n})$. The Künneth isomorphism is not compatible with the actions of $\mathfrak{S}_n$, but we rather have

$$\kappa \circ \sigma = \varepsilon(\sigma) \cdot (\sigma \circ \kappa)$$

for $\sigma \in \mathfrak{S}_n$. In particular, $\kappa$ sends the $\varepsilon$-eigenspace in $H^1(\mathbb{A}^1, f)^{\otimes n}$ to the space of invariants, and we can thus identify the $n$-fold exterior power of $H^1(\mathbb{A}^1, f)$ with

$$\bigwedge^n H^1(\mathbb{A}^1, f) = H^n(\mathbb{A}^n, f^{\boxplus n})^{\mathfrak{S}_n} \tag{12.3.1.2}$$

where on the right hand side we really mean invariants. If we look at the action of $\mathfrak{S}_n$ as a $\mathbb{Q}[\mathfrak{S}_n]$-module structure, the space of invariants is the image of the projector

$$\frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\sigma$$

seen as an idempotent endomorphism of the motive $H^n(\mathbb{A}^n, f^{\boxplus n})$.

Theorem 12.3.2. — *Let $n \geqslant 1$ be an integer and $f \in k[x]$ a polynomial of degree $n+1$ with leading term $a$. Define numbers $b, c \in k$ by*

$$b = \sum_{f'(\alpha)=0} f(\alpha), \qquad c = \begin{cases} (-1)^{\frac{n(n-1)}{2}}\frac{2a}{n+1} & \text{if } n \text{ is odd} \\ (-1)^{\frac{n(n-1)}{2}} & \text{if } n \text{ is even,} \end{cases}$$

*where the sum runs over all $\alpha \in \mathbb{C}$ with $f'(\alpha) = 0$ counted with multiplicity. Let $M(\sqrt{c})$ be the one-dimensional Artin motive with period $\sqrt{c}$, as in* (12.2.2.1)*, $M(\sqrt{\pi}) = H^1(\mathbb{A}^1, x^2)$ and $E(b) = H^0(\operatorname{Spec} k, b)$. There is an isomorphism of exponential motives over $k$*

$$\det H^1(\mathbb{A}^1, f) \simeq M(\sqrt{c}) \otimes M(\sqrt{\pi})^{\otimes n} \otimes E(b).$$

12.3.3. — The following proof of Theorem 12.3.2 is in large parts copied from[2] [**BE00**, §5]. We write the polynomial $f \in k[x]$ as

$$f(x) = a_{n+1}x^{n+1} + a_n x^n + \cdots + a_1 x + a_0$$

with $a = a_{n+1} \neq 0$. Since, for any $u \in k^*$, the motives $M(\sqrt{u}) \otimes M(\sqrt{\pi})$ and $H^1(\mathbb{A}^1, ux^2)$ are isomorphic, the theorem claims that there is an isomorphism of exponential motives over $k$

$$\det H^1(\mathbb{A}^1, f) \simeq H^n(\mathbb{A}^n, q),$$

where $q(x_1, \ldots, x_n) = b + cx_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2$.

12.3.4. — The symmetric group $\mathfrak{S}_n$ acts on $\mathbb{A}^n_x := \operatorname{Spec}(k[x_1, \ldots, x_n])$, and leaves the function $f^{\boxplus n}(\underline{x}) = f(x_1) + f(x_2) + \cdots + f(x_n)$ invariant. We start with writing down the quotient variety and the induced function on it. For $1 \leqslant i \leqslant n$, let us write $S_i(\underline{x})$ for the $i$-th symmetric polynomial in the variables $x_1 \ldots x_n$, so $S_1(\underline{x}) = x_1 + \cdots + x_n$, $S_2(\underline{x}) = x_1x_2 + x_1x_3 + \cdots$ and so on up to $S_n(\underline{x}) = x_1x_2\cdots x_n$. Let $s_1, \ldots, s_n$ denote another set of indeterminates. The morphism of affine varieties

$$\mathbb{A}^n_x = \operatorname{Spec}(k[x_1, \ldots, x_n]) \xrightarrow{\ \pi\ } \mathbb{A}^n_s = \operatorname{Spec}(k[s_1, \ldots, s_n])$$

given by the algebra morphism $s_i \longmapsto S_i(\underline{x})$ identifies $\mathbb{A}^n_s$ as the quotient $\mathbb{A}^n_x/\mathfrak{S}_n$. Since $f^{\boxplus n}$ is a symmetric polynomial we have $f^{\boxplus n} = F \circ \pi$ for some unique $F \in k[\underline{s}]$. The morphism $\pi$ induces a morphism of motives

$$H^n(\mathbb{A}^n_s, F) \xrightarrow{\ \pi^*\ } H^n(\mathbb{A}^n_x, f^{\boxplus n})^{\mathfrak{S}_n} = \det H^n(\mathbb{A}^1, f)$$

[2] To ease the comparison with *loc.cit.*, notice that Bloch and Esnault consider connections given by $\nabla(1) = df$, so what they call $f$ is our $-f$.

which will eventually turn ou to be an isomorphism. The key part of the proof is now to produce an automorphism of $\mathbb{A}^n_s$, that is, a change of variables, which turns $F$ into a quadratic form.

12.3.5. — For each integer $i \geqslant 0$, consider the Newton polynomial $P_i(\underline{x}) = x_1^i + x_2^i + \ldots + x_n^i$. Each of the $P_i$ can be written in a unique way as a polynomial in the elementary symmetric polynomials $S_i$. Let us define $Q_i \in k[\underline{s}]$ by

$$Q_i(S_1(\underline{x}), S_2(\underline{x}), \ldots, S_n(\underline{x})) = P_i(\underline{x}),$$

so that we have

$$Q_0(\underline{s}) = n, \qquad Q_1(\underline{s}) = s_1, \qquad Q_2(\underline{s}) = s_1^2 - 2s_2, \qquad Q_3(\underline{s}) = s_1^3 - 3s_1s_2 + 3s_3$$

and, in general,

$$Q_i(\underline{s}) = \sum_{\substack{r_1+2r_2+\ldots+ir_i=i \\ r_1,\ldots,r_i \geqslant 0}} (-1)^i \frac{i(r_1 + \cdots + r_i - 1)!}{r_1! \cdots r_i!} \prod_{j=1}^{i} (-s_j)^{r_j}.$$

The polynomial $Q_i$ has degree $i$ and only contains the variables $s_1, \ldots, s_i$. We do not add a variable $s_{n+1}$ to $Q_{n+1}$. If we declare that $s_i$ has weighted degree $i$, then $Q_i$ is homogeneous of weighted degree $i$. For $i \geqslant 1$ the polynomial $Q_i$ has no constant part, and the linear part of $Q_i$ is $s_i$ for $1 \leqslant i \leqslant n$ and zero otherwise. For $k+l = i$, the monomial $s_k s_l$ appears in $Q_i$ with coefficient $(-1)^i i$ if $k \neq l$, and with coefficient $\frac{1}{2}i$ if $k = l$.

12.3.6. — Let us express the numbers $b$ and $c$ in the statement of Theorem 12.3.2 in terms of the coefficients of $f$ and $F$. The polynomial $f^{\boxplus n} \in k[\underline{x}]$ is the polynomial $f^{\boxplus n}(\underline{x}) = a_0 P_0(\underline{x}) + a_1 P_1(\underline{x}) + \cdots + a_n P_n(\underline{x}) + a_{n+1} P_{n+1}(\underline{x})$, hence

$$F(\underline{s}) = a_0 Q_0(\underline{s}) + a_1 Q_1(\underline{s}) + \cdots + a_n Q_n(\underline{s}) + a_{n+1} Q_{n+1}(\underline{s})$$

by definition. Setting

$$\underline{a} = \tfrac{1}{(n+1)a_{n+1}} \big( -na_n, (n-1)a_{n-1}, \ldots, (-1)^n a_1 \big)$$

we have $b = F(\underline{a})$ by straightforward computation. The constant term of $F$ is $c = na_0$, its linear homogeneous part is $a_1 s_1 + 2a_2 s_2 + \cdots + na_n s_n$. In the homogeneous quadratic part of $F$, we find the terms $s_k s_l$ appear with coefficient $(-1)^i i a_i$ for $k + l = i$ and $k \neq l$, and with coefficient $\frac{1}{2} i a_i$ for $k = l$. If we think of the homogeneous quadratic part of $g$ as the quadratic form associated with a symmetric bilinear form, then the matrix of this form is

$$B = (\nabla^2 F)(0) = \frac{1}{2} \begin{pmatrix} 2a_2 & -3a_3 & \cdots & (-1)^n na_n & (-1)^{n+1}(n+1)a_{n+1} \\ -3a_3 & 4a_4 & \cdots & (-1)^{n+1}(n+1)a_{n+1} & 0 \\ \vdots & \vdots & & & \vdots \\ (-1)^n na_n & (-1)^{n+1}(n+1)a_{n+1} & & 0 & 0 \\ (-1)^{n+1}(n+1)a_{n+1} & 0 & \cdots & 0 & 0 \end{pmatrix}$$

and we notice that its determinant is equal to

$$\det B = (-1)^{\frac{n(n-1)}{2}} \big( \tfrac{n+1}{2} a_{n+1} \big)^n.$$

The sign $(-1)^{\frac{n(n-1)}{2}}$ comes from taking the product of the antidiagonal entries, while the signs we pick up from the matrix entries themselves cancel to $(-1)^{(n-1)n} = 1$. In particular, viewed modulo squares, $\det(B)$ takes the value

$$\det(B) = \begin{cases} (-1)^{\frac{n(n-1)}{2}} \in k^*/k^{*2} & \text{if } n \text{ is even,} \\ (-1)^{\frac{n(n-1)}{2}} \frac{2a_{n+1}}{n+1} \in k^*/k^{*2} & \text{if } n \text{ is odd,} \end{cases}$$

or $\det(B) = c$ for short, with the notations of Theorem 12.3.2.

LEMMA 12.3.7. — *The differential form $dF$ on $\mathbb{A}^n_s$ (or equivalently, the gradient $\nabla F$ of $F$) vanishes at the point $\underline{a}$ and nowhere else.*

PROOF. □

PROOF OF THEOREM 12.3.2. We start with an affine change of variables, setting $G(\underline{s}) := F(\underline{s}+\underline{a}) - b$. The polynomial $G(\underline{s})$ satisfies $G(0) = 0$ and its gradient $\nabla G$ only vanishes at $0 \in \mathbb{A}^n_s$. Thus, $G$ contains no constant and no linear terms, and we may write $G$ uniquely as

$$G(\underline{s}) = Q(\underline{s}) + R(\underline{s}) + H(\underline{s})$$

where $Q$ and $R$ are homogeneous quadratic polynomials, $Q$ containing the monomials of weight $n+1$ and $R$ containing monomials of weight $\leqslant n$ and each monomial in $H$ has degree $\geqslant 3$. This makes sense, since indeed all monomials in $F$ are of weight $\leqslant n+1$, hence all monomials in $G$, $Q$, $R$ and $H$ are so too. If a monomial of highest possible weight $n+1$ appears in $F$, then the same monomial appears in $G$, with the same coefficient. In particular, the matrix form of $(\nabla^2 G)(0)$ is upper left triangular, with the same (non-zero!) antidiagonal coefficients as $B$. In other words, we have

$$Q(\underline{s}) = \lambda \sum_{i=1}^{n} s_i s_{n+1-i}$$

with $\lambda := (-1)^{n+1}(n+1)a_{n+1}$. We will show that there exists, and in fact construct, an automorphism $\Phi : k[\underline{s}] \to k[\underline{s}]$ such that

$$\Phi(G(\underline{s})) = Q(\underline{s}) \tag{12.3.7.1}$$

holds. To do so, we prove by induction on $j \geqslant 1$ the following:

**Claim:** *There exists an automorphism $\Phi : k[\underline{s}] \to k[\underline{s}]$ such that $\Phi(G) \in k[\underline{s}]$ has the form*

$$\Phi(G(\underline{s})) = Q(\underline{s}) + H'(\underline{s})$$

*where $H' \in k[\underline{s}]$ is a polynomial in the variables $s_j, s_{j+1}, \dots$ where all monomials are of degree $\geqslant 3$ and of weight $\leqslant n+1$.*

For $j = 1$, a linear unipotent automorphism does the job of $\Phi$. Indeed, setting $\Phi(s_i) = s_i + L_i(\underline{s})$ where $L_i$ is a suitable linear polynomial in the variables $s_1, \dots, s_{i-1}$ yields $\Phi(Q(\underline{s}) + R(\underline{s})) = Q(\underline{s})$, hence

$$\Phi(G(\underline{s})) = Q(\underline{s}) + H'(\underline{s})$$

where all monomials in $H'$ are of degree $\geqslant 3$ and of weight $\leqslant n+1$. Now fix $j \geqslant 1$, and suppose that we have found an automorphism $\Phi$ of $k[\underline{s}]$ satisfying the conditions in the claim. The monomial of lowest weight which can possibly occur in $H'$ is $s_j^3$. Hence if $j > \frac{n+1}{3}$, then $H' = 0$ and we are done. Let us suppose thus that $j < \frac{n+1}{2}$. The variable $s_{n+1-j}$ does not appear in $H'(\underline{s})$ again for weight reasons. Indeed, if $s_{n+1-j}$ appears in a monomial of $H'$, then this monomial must have degree $\geqslant 3$, hence would have weight at least $(n+1-j)+2j > n+1$. Let us write

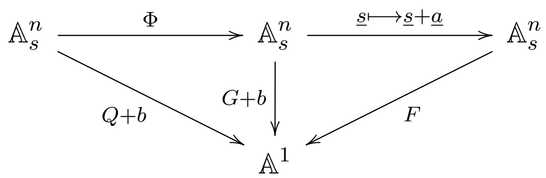
$$H'(s_j, \ldots, s_{n-j}) = H'(0, s_{j+1}, \ldots, s_{n-j}) + s_j \psi(s_j, \ldots, s_{n-j})$$

and define an automorphism $\Psi$ of $k[\underline{s}]$ by $\Psi(s_{n+1-j}) = s_{n+1-j} - (2\lambda)^{-1}\psi(s_j, \ldots, s_{n-j})$ and $\Psi(s_i) = s_i$ for $i \neq n+1-j$. We notice that monomials in $\psi$ have degree at least 2. We find

$$\begin{aligned} \Psi(\Phi(G(\underline{s}))) &= \Psi(\Phi(Q(\underline{s}))) + \Psi(\Phi(R(\underline{s}))) + \Phi(H(\underline{s})) \\ &= \Phi(Q(\underline{s})) + -s_j\psi(\underline{s}) + H'(0, s_{j+1}, \ldots, s_{n-j}) + s_j(\underline{s}) \\ &= Q(\underline{s}) + H''(\underline{s}) \end{aligned}$$

where $H''(\underline{s}) = H'(0, s_{j+1}, \ldots, s_n)$ has the property that all of its terms are of degree $\geqslant 3$ and weight $\leqslant n+1$. The composite $\Psi \circ \Phi$ satisfies thus the property of the claim for $j+1$.

Let us now fix an automorphism $\Phi$ of $k[\underline{s}]$ satisfying (12.3.7.1) and interpret it as an automorphism of $\mathbb{A}_s^n = \operatorname{Spec} k[\underline{s}]$. The diagram




commutes, hence induces the sought after isomorphism of motives

$$H^n(\mathbb{A}_s^n, F) \to H^n(\mathbb{A}_s^n, Q+b). \qquad \square$$

Corollary 12.3.8. — *Let $G \subseteq \mathrm{GL}_n$ be the motivic Galois group of $H^1(\mathbb{A}^1, f)$. The determinant induces a surjective group morphism* $\det\colon G \to \mathbb{G}_m$.

Proof. The determinant $\det H^1(\mathbb{A}^1, f)$ is a rank one object of the category $\mathbf{M}^{\exp}(k)$, hence its motivic Galois group is either $\mathbb{G}_m$ or a group of roots of unity. To exclude the second case, we observe that the isomorphism in Theorem 12.3.2 implies that no tensor power of $\det H^1(\mathbb{A}^1, f)$ becomes the unit object, for example because this motive has weight $n \geqslant 1$. $\square$

Corollary 12.3.9. — *We keep the notation from Theorem 12.3.2. Up to multiplication by a non-zero element of $k$, the determinant of a period matrix of the motive $H^1(\mathbb{A}^1, f)$ is equal to*

$$\sqrt{c} \cdot \pi^{\frac{n}{2}} \cdot e^b. \tag{12.3.9.1}$$

Example 12.3.10. — Let $d \geqslant 2$ be an integer and $f = x^d$. According to Example 1.1.4 from the introduction, the period matrix of the exponential motive $H^1(\mathbb{A}^1, x^d)$ with respect to suitable bases of the de Rham and Betti realisations reads

$$P = \left( \frac{\xi^{ab}-1}{d} \Gamma\left(\frac{a}{d}\right) \right)_{1 \leqslant a, b \leqslant d-1}.$$

Therefore, viewed as an element of $\mathbb{C}^\times/\mathbb{Q}^\times$, the determinant is equal to

$$\det P = \frac{\det(\xi^{ab}-1)}{d^{d-1}} \prod_{a=1}^{d-1} \Gamma\left(\frac{a}{d}\right).$$

LEMMA 12.3.11. — *The equality* $\det(\xi^{ab}-1)_{1\leqslant a,b\leqslant d-1} = i^{\frac{(3d-2)(d-1)}{2}} d^{\frac{d}{2}}$ *holds.*

PROOF. Let $\Delta$ denote the determinant on the left-hand side. Subtracting the first column from each other column of the Vandermonde matrix $(\xi^{ab})_{0\leqslant a,b\leqslant d-1}$ yields the expression

$$\Delta = \det(\xi^{ab})_{0\leqslant a,b\leqslant d-1} = \prod_{0\leqslant a<b\leqslant d-1} (\xi^b - \xi^a).$$

Noting that, for fixed $b$, the product $\prod_{a\neq b}(\xi^b-\xi^a)$ is the derivative of the polynomial $x^d-1$ evaluated at $x=\xi^b$, one computes the absolute value

$$|\Delta| = \prod_{0\leqslant a\neq b\leqslant d-1} |\xi^b-\xi^a|^{\frac{1}{2}} = d^{\frac{d}{2}}.$$

We are thus left to determine the argument of $\Delta$. In terms of the notation $\mathbf{e}(x)=\exp(2\pi i x)$, dear to analytic number theorists, one has

$$\xi^b - \xi^a = \mathbf{e}\left(\tfrac{a+b}{2d}\right)\left(\mathbf{e}(\tfrac{b-a}{2d}) - \mathbf{e}(\tfrac{a-b}{2d})\right) = 2\mathbf{e}\left(\tfrac{a+b}{2d}\right) i\sin(\tfrac{\pi(b-a)}{d}),$$

and the sine is positive when $a$ and $b$ satisfy $0\leqslant a<b\leqslant d-1$. Then a straightforward computation allows one to conclude:

$$\frac{\Delta}{|\Delta|} = \prod_{0\leqslant a<b\leqslant d-1} \mathbf{e}\left(\tfrac{a+b}{2d}\right) i = \mathbf{e}\left(\tfrac{(d-1)^2}{4}\right) i^{\frac{d(d-1)}{2}} = i^{\frac{(3d-2)(d-1)}{2}}. \qquad \square$$

REMARK 12.3.12. —

Putting everything together, we get the expression $\det P = d^{1-\frac{d}{2}} i^{\frac{(3d-2)(d-1)}{2}} \prod_{a=1}^{d-1}\Gamma(\frac{a}{d})$. Besides, Corollary 12.3.9 specialises to the equality $\det P = \sqrt{c}\cdot\pi^{\frac{d-1}{2}}$ in $\mathbb{C}^\times/\mathbb{Q}^\times$. Combined with the previous calculation, this implies

$$\prod_{j=0}^{d-1} \Gamma\left(\frac{j}{d}\right) \sim_{\mathbb{Q}^\times} \frac{(2\pi)^{\frac{d-1}{2}}}{\sqrt{d}}, \qquad (12.3.12.1)$$

thus showing that the multiplication formula for the gamma values has motivic origin. Both sides of (12.3.12.1) are actually equal.

12.3.13 (Computation of the epsilon factor). — Let $F=\mathbb{F}_q$ be a finite field with $q$ elements. Given a smooth variety $X$ over $F$ and a $\overline{\mathbb{Q}}_\ell$-sheaf $\mathscr{F}$ on $X$, the *epsilon factor* is defined as

$$\varepsilon(X,\mathscr{F}) = \prod_{j\geqslant 0} \det(-\varphi_F \mid H^j_c(X_{\overline{F}},\mathscr{F}))^{(-1)^j} \in \overline{\mathbb{Q}}_\ell^\times,$$

where $\varphi_F$ stands for the geometric Frobenius.

THEOREM 12.3.14. — *Let $p$ be a prime number and $f \in \mathbb{F}_p[x]$ a polynomial of degree $n+1$.*

$$\varepsilon(\mathbb{A}^1_{\mathbb{F}_p}, f^*\mathcal{L}_\psi) = \begin{cases} \psi(b)q^{\frac{n}{2}} & n \text{ is even} \\ -\psi(b)G(\chi_2,\psi)\chi_2(c)q^{\frac{n-1}{2}} & n \text{ is odd.} \end{cases}$$

## 12.4. Bessel motives

We have already encountered in the introduction, Example 1.1.5, a two-dimensional exponential motive whose periods are special values of the modified Bessel functions. Namely, one considers the variety $\mathbb{G}_m = \operatorname{Spec} \overline{\mathbb{Q}}[x, x^{-1}]$ and the function $f_\lambda = -\frac{\lambda}{2}(x - \frac{1}{x})$, where $\lambda$ is a non-zero algebraic number, say a non-zero element of a number field $k \subseteq \mathbb{C}$. The *Bessel motive* associated to $\lambda$ is

$$B(\lambda) = H^1(\mathbb{G}_m, f_\lambda)(1)$$

seen as an object of $\mathbf{M}^{\exp}(k)$. It is a two-dimensional motive. The rapid decay homology $H_1(X, f_\lambda)$ has a basis consisting of a simple loop around 0 and a path joining the two connected components of $f_\lambda^{-1}(S_r)$ for large $r > 0$. Having chosen such a basis, we can identify the motivic Galois group of $B(\lambda)$ with a closed subgroup of $\mathrm{GL}_2$. We will in this section compute various realisations and the motivic fundamental group of $B(\lambda)$.

PROPOSITION 12.4.1. — $\det B(\lambda) = \mathbb{Q}(1)$

PROOF. The determinant of $B(\lambda)$ is the one-dimensional motive

$$\det B(\lambda) = H^2(X \times X, f \boxplus f)^{\mathfrak{S}_2}(2)$$

Consider the morphism $X \times X \to \mathbb{A}^2$ given by the algebra morphism $\varphi\colon k[s,t] \to k[x, x^{-1}, y, y^{-1}]$ sending $s$ to $x+y$ and $t$ to $(xy)^{-1}$. Setting $g(s,t) = s + \frac{\lambda^2}{4}st$ we have $\varphi(g(s,t)) = f(x) + f(y)$, hence a morphism of motives

$$H^2(\mathbb{A}^2, g) \to H^2(\mathbb{A}^2, f \boxplus f)^{\mathfrak{S}_2} \tag{12.4.1.1}$$

induced by $\varphi$. Since $\frac{\lambda^2}{4}$ is a square in $k^*$, the motive $H^2(\mathbb{A}^2, g)$ is that of the quadratic form $(s,t) \longmapsto st$ which has determinant $-1$, hence $H^2(\mathbb{A}^2, g) = \mathbb{Q}(-1)$. It remains to check that the morphism (12.4.1.1) is non-zero. □

PROPOSITION 12.4.2. — *Let $F[1] = \mathrm{R}_{\mathrm{perv}} B(\lambda)$ be the perverse realisation of $B(\lambda)$. The singularities of the constructible sheaf $F$ are at the points $\{i\lambda, -i\lambda\}$. With respect to an appropriate basis of $F_0$, the local monodromy operators of the local system on $\mathbb{C} \setminus \{\pm i\lambda\}$ defined by $F$ are*

$$\rho_+ = \begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix} \qquad \text{and} \qquad \rho_- = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix},$$

*and the fibres of $F$ at the singular points are the local invariants.*

PROOF. Let $p\colon \mathbb{G}_m \times \mathbb{A}^1 \to \mathbb{A}^1$ be the projection and write $\underline{\mathbb{Q}}$ for the constant sheaf with value $\mathbb{Q}$ on $\mathbb{G}_m \times \mathbb{A}^1$. Let $j$ be the inclusion into $\mathbb{G}_m \times \mathbb{A}^1$ of the complement of the closed subvariety

$$\Gamma = \{(x,z) \mid f_\lambda(x) = z\}$$

of $\mathbb{G}_m \times \mathbb{A}^1$. The sheaf $F$ is $R^1p_*(j_!j^*\underline{\mathbb{Q}})$. Rewriting the equation $f_\lambda(x) = z$ as

$$x^2 + \tfrac{2z}{\lambda}x - 1 = 0$$

shows that the singularities of $F$ are located at those points $z \in \mathbb{C}$ where the discriminant of the quadratic polynomial $x^2 + \frac{2z}{\lambda}x - 1$ vanishes, and this discriminant equals $4(z^2\lambda^{-2}+1)$. In order to compute the monodromy of $F$ around the singularities $\pm i\lambda$, consider the basis of

$$V_0 := F_0^\vee = H_1(\mathbb{C}^\times, \{\pm 1\})$$

given by a standard loop $\varphi$ around 0, and the sum $\gamma = \gamma_+ + \gamma_-$, where $\gamma_+$ is an arc from $-1$ to 1 in the upper half plane, and $\gamma_-$ is an arc from $-1$ to 1 in the lower half plane. As $z$ runs over a loop $\rho_+$ around $i\lambda$, say

$$\rho_+ : t \longmapsto i\lambda + \lambda e^{2\pi i(t-1/4)}$$

the roots of the polynomial

$$x^2 + \tfrac{2z}{\lambda}x - 1 = x^2 + 2(i + e^{2\pi i(t-1/4)})x - 1$$

exchange positions, moving in the lower half plane. The monodromy action $\rho_+$ is accordingly given by $\rho_+(\varphi) = \varphi$, $\rho_+(\gamma_+) = -\gamma_- - \varphi$ and $\rho_+(\gamma_-) = -\gamma_-$. With respect to the basis $\varphi, \gamma$, the monodromy operator for the loop $\rho_+$ acts on $V_0$ as the matrix

$$\left(\begin{smallmatrix} 1 & -2 \\ 0 & -1 \end{smallmatrix}\right)$$

so the matrix of $\rho_+$ on the dual space $F_0 = V_0^\vee$ is given by the transposed matrix. The computation of the matrix of $\rho_-$ with respect to the same basis is similar. Finally, since $F$ is an object of $\mathbf{Perv}_0$, the dimensions of the fibres $\dim F_{i\lambda}$ and $\dim F_{-i\lambda}$ must add up to $2 = \dim F_0$, hence must consist of all the local invariants. In terms of the basis dual to $\varphi, \gamma$, the invariants are the one dimensional subspaces generated by the vectors $\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)$ for $\rho_+$ and by $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ for $\rho_-$. □

PROPOSITION 12.4.3. — *The motivic Galois group of the Bessel motive $B(\lambda)$ is* $\mathrm{GL}_2$.

PROOF. Let $G \subseteq \mathrm{GL}_2$ denote the motivic Galois group of $B(\lambda)$ and let $H \subseteq G$ be the tannakian fundamental group of the perverse realisation $F[1] = \mathrm{R}_{\mathrm{perv}}B(\lambda)$ of $B(\lambda)$. We first notice that $H$ and $G$ are both reductive. Indeed, the perverse sheaf $F[1]$ is a simple object in the category $\mathbf{Perv}_0$ since already the local system defined by $F$ is simple. In follows that $B(\lambda)$ itself is simple too, and in any tannakian category of characteristic zero the fundamental group of any simple or semisimple object is reductive. By Proposition 12.4.1, the group $G$ surjects to $\mathbb{G}_m$ via the determinant map, and since the perverse realisation of $\mathbb{Q}(1)$ is trivial, the group $H$ is contained in $\mathrm{SL}_2$. Again, since $F[1]$ is simple, the tautological two-dimensional representation of $H$ as a subgroup of $\mathrm{SL}_2$ is irreducible, but the only algebraic subgroup of $\mathrm{SL}_2$ with this property is $\mathrm{SL}_2$ itself. It follows that $H = \mathrm{SL}_2$ and $G = \mathrm{GL}_2$ as claimed. □

12.4.4. — As a consequence of Proposition 12.4.3, the period conjecture specialises to the following statement:

CONJECTURE 12.4.5. — *For every non-zero algebraic number $\lambda \in \mathbb{C}$, the following complex numbers are algebraically independent:*

$$I_0(\lambda),\ I_1(\lambda),\ \tfrac{1}{2\pi i}K_0(\lambda),\ \tfrac{1}{2\pi i}K_1(\lambda).$$

## 12.5. Special values of $E$-functions

DEFINITION 12.5.1 (Siegel). — Let $f$ be an entire function given by a power series

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$$

with algebraic coefficients $a_n$. For each $n \geqslant 1$, let $||a_n||$ denote the largest absolute value of all complex conjugates of $a_n$, and let $d_n \in \mathbb{Z}_{\geqslant 1}$ be the smallest integer such that $d_n a_1, d_n a_2, \ldots, d_n a_n$ are all algebraic integers. The function $f$ is called an *$E$-function* if

- it satisfies a linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$,
- there exists a constant $C > 0$ such that $\|a_n\| \leqslant C^n$ and $d_n \leqslant C^n$ for all $n \geqslant 1$.

12.5.2. — Instead of $d_n \leqslant C^n$, Siegel [**Sie29**] asks for the seemingly less stringent condition that, for every $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$ such that $d_n \leqslant C_\varepsilon (n!)^\varepsilon$ holds for all $n$. However, no examples of functions satisfying the latter condition but not the former one are known[3]. An elegant alternative way to formulate the growth condition on the coefficients is to ask for

$$h([a_0 : a_1 : a_2 : \cdots : a_n]) = O(n)$$

where $h$ stands for logarithmic height on $\mathbb{P}^n$. Standard examples of $E$-functions include the exponential function, and the Bessel function $J_0(z^2)$. Polynomials are $E$-functions too. The exponential integral functions $E_n$ are not $E$-functions, already because they have a singularity at 0.

THEOREM 12.5.3 (Siegel–Shidlovskii). — *Let $f = (f_1, \ldots, f_n)$ be $E$-functions which satisfy the linear differential equation $f = Af'$ for some $n$ by $n$ matrix with coefficients in $\overline{\mathbb{Q}}(z)$. The equality*

$$\mathrm{trdeg}_{\mathbb{Q}}(f_1(\alpha), \ldots, f_n(\alpha)) = \mathrm{trdeg}_{\mathbb{C}(z)}(f_1(z), \ldots, f_n(z))$$

*holds for any non-zero $\alpha \in \overline{\mathbb{Q}}$ which is not a pole of any of the coefficients of $A$.*

[3]Since $f$ satisfies a differential equation, which can be thought of as a kind of linear recurrence relation for the coefficients $\alpha$, Siegel's condition should imply the one we gave in the definition.

## 12.6. Special values of exponential integral functions

In this section, we introduce exponential motives whose periods contain special values of the exponential integral functions $E_1$. The theorem of Siegel–Shidlovskii about special values of $E$-functions shows that a small part of the period conjecture holds for these motives.

12.6.1. — Recall that, for each integer $n$, the exponential integral function $E_n$ is defined, in the half-plane $\mathrm{Re}(s) > 0$, by the convergent integral

$$E_n(s) = \int_1^\infty e^{-sx} \frac{dx}{x^n}.$$

In particular, $E_0(s) = \frac{e^{-s}}{s}$. As a function of $s$, this integral defines a holomorphic function on the right half complex plane, which extends to a holomorphic function on $\mathbb{C} \setminus [-\infty, 0]$. The function $E_n$ is closely related to the incomplete gamma function

$$\Gamma(s, x) = \int_x^\infty t^{s-1} e^{-t} dt,$$

namely by $E_n(s) = s^{n-1}\Gamma(n-1, s)$. Integration by parts shows the recurrence relation

$$nE_{n+1}(s) = e^{-s} - sE_n(s)$$

which allows us to calculate $E_n$ for $n \leqslant 0$ from $E_0$, and $E_n$ for $n \geqslant 1$ from $E_1$. In particular we see that for $n \leqslant 0$ the function $E_n(s)$ is a rational function of $e^s$ and $s$, whereas for $n \geqslant 1$ the function $E_n(s)$ is a rational function of $s$, $e^s$ and $E_1(s)$. One can show that the field extension of $\mathbb{C}(s)$ generated by $\{E_n(s) \mid n \in \mathbb{Z}\}$ has transcendence degree 2. In other words, the functions $e^s$ and $E_1(s)$ are algebraically independent over $\mathbb{C}(s)$.

12.6.2. — Special values of exponential integral functions $E_n$ are not very much studied, with the notable exception of the so-called *Gompertz constant*

$$G = e \cdot E_1(1)$$

which admits an intriguing continued fraction representation, due to Stieltjes.

12.6.3. — Let $\alpha$ be a non-zero algebraic number and set $k = \mathbb{Q}(\alpha)$. The integral representation suggests to look at the following exponential motive over $k$:

$$M = H^1(\mathbb{G}_m, \{1\}, \alpha x).$$

By the exact sequence for triples (4.2.4.2), $M$ fits into an exact sequence

$$0 \longrightarrow E(-\alpha) \longrightarrow M \longrightarrow H^1(\mathbb{G}_m, \alpha x) \longrightarrow 0.$$

We claim that $H^1(\mathbb{G}_m, \alpha x)$ is isomorphic to $H^1(\mathbb{G}_m)$ as exponential motive. To see this, consider $X = \mathbb{G}_m \times \mathbb{A}^1 = \operatorname{Spec} k[x, x^{-1}, t]$, together with the function $f(x,t) = xt$. The inclusions of $\mathbb{G}_m$ into $X$ as $\mathbb{G}_m \times \{0\}$, resp. as $\mathbb{G}_m \times \{\alpha\}$, yield morphisms of motives $H^1(\mathbb{A}^1, f) \to H^1(\mathbb{G}_m)$, resp. $H^1(\mathbb{A}^1, f) \to H^1(\mathbb{G}_m, \alpha x)$, which are isomorphisms. Therefore, we get an extension

$$0 \longrightarrow E(-\alpha) \longrightarrow M \longrightarrow \mathbb{Q}(-1) \longrightarrow 0.$$

With respect to these bases, the period matrix reads:

| $\int$ | | |
|---|---|---|
| | $e^{-\alpha}$ | $E_1(\alpha)$ |
| | $0$ | $2\pi i$ |

LEMMA 12.6.4. — *The power series* $f(z) := \sum_{n=1}^{\infty} \frac{1}{n \cdot n!} z^n$ *is a transcendental E-function.*

PROOF. It is clear that $f(z)$ is an entire function, and the coefficients $a_n = \frac{1}{n}$ are bounded. We only have to check that for some constant $C$, the inequality

$$d_n := \operatorname{lcm}(1, 2, 3, 4, 5, \ldots, n) \leqslant C^n$$

holds. This lowest common multiple is conveniently expressed using the summatory von-Mangoldt function or secondary Chebyshev function $\psi(n)$: We have

$$d_n = \exp(\psi(n))$$

for all $n \geqslant 1$. The function $\psi$ grows asymptotically as $\psi(x) \sim x$ - this is equivalent to the prime number theorem. In particular $\psi(x) < cx$ for sufficiently large $c > 1$, hence $d_n < e^{cn} = C^n$. In order to show that $f$ is a $E$-function, it remains to find a linear differential equation for $f$. Indeed, we have $1 + zf'(z) = e^z$ by inspection of the power series, hence

$$(1 + zf'(z))' = 1 + zf'(z)$$

is the differential equation $(e^z)' = e^z$. Rearranging terms yields the differential equation

$$zf''(z) + (1 - z)f'(z) = 1$$

which is only affine and not linear, but we can always derive once more. □

The general solution of $zu'' + (1 - z)u' = 1$ is $A + Bf(z) - \log(z)$ for constants $A$ and $B$. Unfortunately, $\log(z)$ is not an $E$-function. The Siegel–Shidlovski theorem thus only shows that special values of the function $f$, such as

$$f(-1) = \int_0^1 \int_0^1 e^{-xy} dx dy$$

are transcendental. Once we show that $f(z)$ and $e^z$ are algebraically independent over $\mathbb{C}(z)$, we will obtain algebraic independence over $\overline{\mathbb{Q}}$ of, say, $f(-1)$ and $e$.

## 12.7. Laurent polynomials and special values of $E$-functions

We fix a number field $k \subseteq \mathbb{C}$. Regular functions on $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$ are Laurent polynomials with coefficients in $k$, so we obtain a motive $M = H^1(\mathbb{G}_m, f)$ from every Laurent polynomial $f$. In this section, we show how to relate some of the periods of $M$ to special values of $E$-functions. The Siegel-Shidlovskii theorem allows us then to prove some transcendence results.

12.7.1. — Let $f \in k[x, x^{-1}]$ be a Laurent polynomial of the form

$$f(x) = \frac{1}{d}(c_{-r}x^{-r} + \cdots + c_s x^s)$$

where $d > 0$ is an integer and the $c_i \in \mathcal{O}_k$ are algebraic integers. We assume that $r$ and $s$ are both positive and the coefficients $c_{-r}$ and $c_s$ non-zero. The motive $M = H^1(\mathbb{G}_m, f)$ has dimension $r+s$. A particular element in the rapid decay homology of $(\mathbb{G}_m, f)$ is the standard loop $\gamma$ winding once counterclockwise around 0. Given another Laurent polynomial $g \in k[x, x^{-1}]$, we set

$$E(g, z) = \frac{1}{2\pi i}\oint e^{-zf(x)}g(x)dx,$$

where $z$ is a complex variable and the integral sign means integration along the loop $\gamma$. The function $E(g, z)$ is entire and satisfies the following three relations:

$$aE(g, z) + bE(h, z) = E(ag + bh, z), \tag{12.7.1.1}$$

$$\tfrac{\partial}{\partial z}E(g, z) = -E(fg, z), \tag{12.7.1.2}$$

$$E(g', z) = zE(f'g, z). \tag{12.7.1.3}$$

In the first one, $a$ and $b$ are scalars in $k$ and $h$ is another Laurent polynomial. The second one is obtained by differentiating under the integral sign, which is allowed since the cycle $\gamma$ is compact. Finally, the third one follows from Stokes' formula and could be rewritten as $E(d_{zf}(g), z) = 0$.

Proposition 12.7.2. — *The function $E(g, z)$ is an $E$-function.*

Proof. We have to verify that $E(g, z)$ satisfies a non-zero linear differential equation, and that the coefficients $a_n$ of the Taylor expansion

$$E(g, z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$$

lie in a common number field and their logarithmic height has at most linear growth. By (12.7.1.1) and the fact that linear combinations of $E$-functions are again $E$-functions, we may assume that $g$ is a monomial, say $g(x) = x^d$ for some integer $d \in \mathbb{Z}$.

Let us start with bounding the coefficients. By (12.7.1.2), they are equal to

$$a_n = (-1)^n E(f^n g, 0) = \frac{(-1)^n}{2\pi i}\oint f(x)^n g(x)dx$$

which is, by Cauchy's formula, the coefficient of $x^{-1}$ in the Laurent polynomial $f(x)^n g(x)$. Since we already assume $g(x) = x^d$, the coefficient $a_n$ is the coefficient of $x^{1-d}$ in $f(x)^n$. It is thus clear that $a_n$ belongs to $k$. Moreover, we can write $a_n$ in terms of the coefficients of $f$ as

$$a_n = \frac{1}{d^n} \sum c_{i_1} c_{i_2} \cdots c_{i_n},$$

where the sum runs over all $n$-tuples of integers $(i_1, \ldots, i_n) \in [-r, s]^n$ satisfying $i_1 + \cdots + i_n = 1 - d$. Define $C := \max\{\|c_{-r}\|, \ldots, \|c_n\|\}$. The estimate

$$\|a_n\| \leqslant (r+s)^n \max\left\{\|c_{i_1} c_{i_2} \cdots c_{i_n}\| \;\middle|\; -r \leqslant i_1, \ldots, i_n \leqslant s\right\} \leqslant (r+s)^n \cdot C^n$$

is what was needed in Definition 12.5.1. It remains to show that $E(g, z)$ satisfies a non-trivial differential equation. This is a straightforward consequence of the relations (12.7.1.1), (12.7.1.2) and (12.7.1.3). Indeed, the functional equation tells us that the $\mathbb{C}(z)$-linear space of entire functions spanned by $\{E(g, z) \mid g \in k[x, x^{-1}]\}$ is finite-dimensional, of dimension at most $r + s$. Therefore, for any fixed $g$, the functions $\frac{\partial^i}{\partial z^i}\{E(g, z) \mid 0 \leqslant i \leqslant r + s\}$ are $\mathbb{C}(z)$-linearly dependent. □

12.7.3. — In the proof of proposition 12.7.2, we have explained why the function $E(g, z)$ satisfies a differential equation of order $\leqslant r + s$. Let us now describe an explicit construction of this differential equation, in the form of a system of first order linear differential equations. Set $E_p(z) = E(x^p, z)$. Our goal is to produce an equation

$$\tfrac{\partial}{\partial z} E = LE$$

where $E$ is the vector of functions $(E_0, \ldots, E_{r+s-1})$ and $L$ is a matrix with coefficients in $k(z)$. The functional equation (12.7.1.3) applied to $g(x) = x^{p+1}$ reads

$$(p+1)E_p(z) = zE(x^p f'(x), z) = \frac{z}{d} \sum_{q=-r}^{s} q c_q E_{p+q}(z) \tag{12.7.3.1}$$

For each $p \in \mathbb{Z}$, we can determine uniquely $a_{pq} \in k[z, z^{-1}]$ such that

$$E_p(z) = \sum_{q=0}^{r+s-1} a_{pq}(z) E_q(z) \tag{12.7.3.2}$$

holds. This is indeed possible, trivially so for $0 \leqslant p < r + s$ and inductively on $p$ for $p < 0$ and $p \geqslant r + s$. For our needs, we need to determine the coefficients $a_{pq}$ for $-r \leqslant p < 0$ and $r + s \leqslant p < r + 2s$. The differential relation (12.7.1.2) in the case $g(x) = x^p$ reads

$$E_p'(z) = -E(x^p f(x), z) = -\sum_{q=-r}^{s} c_q E_{p+q}(z) \tag{12.7.3.3}$$

For each $0 \leqslant p < r + s$, we can substitute the relations (12.7.3.2) into the right hand side of (12.7.3.3), and obtain so the sought system of differential equations. The coefficients of the matrix $L$ are linear combinations of the $a_{pq} \in k[z, z^{-1}]$, hence are themselves elements of $k[z, z^{-1}]$.

EXAMPLE 12.7.4. — Consider the Laurent polynomial $f(x) = x^{-3} + x^{-1} + x + x^3$. Since $f$ is odd, we expect that the resulting motive $M = H^1(\mathbb{G}_m, f)$ has some extra symmetries. The dimension of $M$ is $3+3-1=5$. The diferential forms $dx, xdx, \dots, x^5dx$ represent a basis of the deRham cohomology $H^1_{\mathrm{dR}}(\mathbb{G}_m, zf)$. Here are the equations (12.7.3.1) for $p = 0,1,2,3,4,5$.

$$\begin{aligned}
E_0(z) &= -3zE_{-3}(z) - zE_{-1}(z) + zE_1(z) + 3zE_3(z)\\
2E_1(z) &= -3zE_{-2}(z) - zE_0(z) + zE_2(z) + 3zE_4(z)\\
3E_2(z) &= -3zE_{-1}(z) - zE_1(z) + zE_3(z) + 3zE_5(z)\\
4E_3(z) &= -3zE_0(z) - zE_2(z) + zE_4(z) + 3zE_6(z)\\
5E_4(z) &= -3zE_1(z) - zE_3(z) + zE_5(z) + 3zE_7(z)\\
6E_5(z) &= -3zE_2(z) - zE_4(z) + zE_6(z) + 3zE_8(z)
\end{aligned}$$

The linear relations (12.7.3.2) for $p = -1,-2,-3$ and $p = 6,7,8$ are obtained from these. Here they are.

$$\begin{aligned}
E_{-1}(z) &= -\tfrac{1}{3}E_1(z) - \tfrac{1}{z}E_2(z) + \tfrac{1}{3}E_3(z) + E_5(z)\\
E_{-2}(z) &= -\tfrac{1}{3}E_0(z) - \tfrac{2}{3z}E_1(z) + \tfrac{1}{3}E_2(z) + E_4(z)\\
E_{-3}(z) &= -\tfrac{1}{3}E_{-1}(z) - \tfrac{1}{3z}E_0(z) + \tfrac{1}{3}E_1(z) + E_3(z)\\
&= -\tfrac{1}{3z}E_0(z) + \tfrac{4}{9}E_1(z) + \tfrac{1}{3z}E_2(z) + \tfrac{8}{9}E_3(z) - \tfrac{1}{3}E_5(z)\\
E_6(z) &= E_0(z) + \tfrac{1}{3}E_2(z) + \tfrac{4}{3z}E_3(z) - \tfrac{1}{3}E_4(z)\\
E_7(z) &= E_1(z) + \tfrac{1}{3}E_3(z) + \tfrac{5}{3z}E_4(z) - \tfrac{1}{3}E_5(z)\\
E_8(z) &= E_2(z) + \tfrac{1}{3}E_4(z) + \tfrac{2}{z}E_5(z) - \tfrac{1}{3}E_6(z)\\
&= -\tfrac{1}{3}E_0(z) + \tfrac{8}{9}E_2(z) - \tfrac{4}{9z}E_3(z) + \tfrac{4}{9}E_4(z) + \tfrac{2}{z}E_5(z)
\end{aligned}$$

Next, let us write the differential relations (12.7.3.3) for $0 \leqslant p < r+s$.

$$\begin{aligned}
E_0'(z) &= -E_{-3}(z) - E_{-1}(z) - E_1(z) - E_3(z)\\
E_1'(z) &= -E_{-2}(z) - E_0(z) - E_2(z) - E_4(z)\\
E_2'(z) &= -E_{-1}(z) - E_1(z) - E_3(z) - E_5(z)\\
E_3'(z) &= -E_0(z) - E_2(z) - E_4(z) - E_6(z)\\
E_4'(z) &= -E_1(z) - E_3(z) - E_5(z) - E_7(z)\\
E_5'(z) &= -E_2(z) - E_4(z) - E_6(z) - E_8(z)
\end{aligned}$$

Substituting $E_{-3}$, $E_{-2}$, $E_{-1}$ and $E_6$, $E_7$, $E_8$, we obtain:

$$\begin{aligned}
E_0'(z) &= \tfrac{1}{3z}E_0(z) - \tfrac{10}{9}E_1(z) + \tfrac{2}{3z}E_2(z) - \tfrac{20}{9}E_3(z) - \tfrac{2}{3}E_5(z)\\
E_1'(z) &= -\tfrac{2}{3}E_0(z) + \tfrac{2}{3z}E_1(z) - \tfrac{4}{3}E_2(z) - 2E_4(z)\\
E_2'(z) &= -\tfrac{2}{3}E_1(z) + \tfrac{1}{z}E_2(z) - \tfrac{4}{3}E_3(z) - 2E_5(z)\\
E_3'(z) &= -2E_0(z) - \tfrac{4}{3}E_2(z) - \tfrac{4}{3z}E_3(z) - \tfrac{2}{3}E_4(z)\\
E_4'(z) &= -2E_1(z) - \tfrac{4}{3}E_3(z) - \tfrac{5}{3z}E_4(z) - \tfrac{2}{3}E_5(z)\\
E_5'(z) &= -\tfrac{2}{3}E_0(z) - \tfrac{20}{9}E_2(z) - \tfrac{8}{9z}E_3(z) - \tfrac{10}{9}E_4(z) - \tfrac{2}{z}E_5(z)
\end{aligned}$$

From this system we can read off the matrix $L$.

$$L = \begin{pmatrix} \frac{1}{3z} & -\frac{10}{9} & \frac{2}{3z} & -\frac{20}{9} & 0 & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3z} & -\frac{4}{3} & 0 & -2 & 0 \\ 0 & -\frac{2}{3} & \frac{1}{z} & -\frac{4}{3} & 0 & -2 \\ -2 & 0 & -\frac{4}{3} & -\frac{4}{3z} & -\frac{2}{3} & 0 \\ 0 & -2 & 0 & -\frac{4}{3} & -\frac{5}{3z} & -\frac{2}{3} \\ -\frac{2}{3} & 0 & -\frac{20}{9} & -\frac{8}{9z} & -\frac{10}{9} & -\frac{2}{z} \end{pmatrix}$$

## 12.8. The Euler–Mascheroni constant

In this section, we describe a two-dimensional exponential motive over $\mathbb{Q}$, extension of $\mathbb{Q}(-1)$ by $\mathbb{Q}(0)$, having the Euler–Mascheroni constant $\gamma$ as one of its periods. That $\gamma$ is an exponential period was observed by Belkale and Brosnan in [**BB03**, Eq. (3.16)]:

$$\gamma = -\int_0^\infty \int_0^1 e^{-x} \frac{x-1}{(x-1)y+1} dydx.$$

Note that, despite the fact that the function $(x-1)y+1$ has a zero at the point $(0,1)$ which lies on the boundary of the integration domain, the integral converges absolutely.

A glance at Wikipedia reveals a number of alternative presentations of $\gamma$ as an exponential period. The one we shall use here is

$$\gamma = -\int_0^\infty \log(x) e^{-x} dx = -\int_0^\infty \int_1^x \frac{1}{y} e^{-x} dydx,$$

which is also an improper but absolutely convergent integral. To get rid of the pole of the integrand, we resort to blowing up the affine plane at the point $(0,0)$. In terms of the integral, this just means that we change variables from $(x,y)$ to $(xy,y)$, thus obtaining:

$$\gamma = -\int_0^\infty \int_1^x \frac{1}{y} e^{-xy} dyd(xy) = \int_0^1 \int_0^1 e^{-xy} dxdy - \int_1^\infty \int_1^\infty e^{-xy} dxdy. \qquad (12.8.0.1)$$

12.8.1 (The Euler-Mascheroni motive). — The integral representation (12.8.0.1) suggests the following geometric picture: let $X = \operatorname{Spec} \mathbb{Q}[x,y]$ be the affine plane, $Y$ the union of four lines given by the equation $xy(x-1)(y-1) = 0$, and $f$ the regular function $f(x,y) = xy$ on $X$. As we will see below, the exponential motive $H^2(X,Y,f)$ turns to be three-dimensional. To get something smaller, we consider the blow-up $\pi\colon \widetilde{X} \to X$ at the point $(1,1)$. Let $\widetilde{Y}$ denote the strict transform of $Y$, $E$ the exceptional divisor, and $\tilde{f} = f \circ \pi$ the induced function on $\widetilde{X}$. The motive $H^2(\widetilde{X}, \widetilde{Y}, \tilde{f})$ is also three-dimensional. The blow-up map $\pi$ yields a rank two morphism of exponential motives

$$\pi^*\colon H^2(X,Y,f) \to H^2(\widetilde{X}, \widetilde{Y}, \tilde{f}).$$

Definition 12.8.2. — The *Euler-Mascheroni motive* $M(\gamma) \subseteq H^2(\widetilde{X}, \widetilde{Y}, \tilde{f})$ is the image of $\pi^*$.

12.8.3 (The motive $M(\gamma)$ as an extension of $\mathbb{Q}(-1)$ by $\mathbb{Q}(0)$). — Let us examine the structure of the motive $M(\gamma)$ in detail. We keep the notation from 12.8.1 and denote by $E$ the exceptional divisor of the blow-up $\pi\colon \widetilde{X} \to X$. Let $Z \subseteq Y$ be the union of two lines defined by $(x-1)(y-1) = 0$, and denote by $\widetilde{Z} \subseteq \widetilde{X}$ the strict transform of $Z$. We consider the following commutative diagram of exponential motives with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & H^1(E, E\cap\widetilde{Y}, 1) & \xrightarrow{\cong} & H^1(E, E\cap\widetilde{Z}, 1) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^1(Y, Z, f|_Y) & \longrightarrow & H^2(X, Y, f) & \longrightarrow & H^2(X, Z, f) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\pi^*} & & \downarrow & & \\
0 & \longrightarrow & H^1(\widetilde{Y}, \widetilde{Z}, \tilde{f}) & \longrightarrow & H^2(\widetilde{X}, \widetilde{Y}, \tilde{f}) & \longrightarrow & H^2(\widetilde{X}, \widetilde{Z}, \tilde{f}) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & H^2(E, E\cap\widetilde{Y}, 1) & \xrightarrow{\cong} & H^2(E, E\cap\widetilde{Z}, 1) & &
\end{array}$$

The middle column comes from the long exact sequence of the pair of immersions $\widetilde{Y} \subseteq \widetilde{Y}\cup E \subseteq \widetilde{X}$ and the excision isomorphism $H^n(\widetilde{Y}\cup E, \widetilde{Y}, \tilde{f}_{|\widetilde{Y}\cup E}) \cong H^n(E, E\cap\widetilde{Y}, 1)$. The right-hand column is obtained in the same way replacing $Y$ by $Z$, and the top and bottom isomorphisms follow from the equality $E\cap\widetilde{Y} = E\cap\widetilde{Z}$. The horizontal short exact sequences are part of the long exact sequences associated with the triples $Z \subseteq Y \subseteq X$ and $\widetilde{Z} \subseteq Y \subseteq X$ respectively. The zeroes on the right-hand side are explained by cohomological dimension. The zeroes on the left-hand side can be obtained by writing out the long exact sequences of the pairs $Z \subseteq X$ and $\widetilde{Z} \subseteq \widetilde{X}$.

PROPOSITION 12.8.4. — *The Euler-Mascheroni motive is a non-split extension of* $\mathbb{Q}(-1)$ *by* $\mathbb{Q}(0)$. *In other words, there is a short exact sequence*

$$0 \to \mathbb{Q}(0) \to M(\gamma) \to \mathbb{Q}(-1) \to 0$$

*of exponential motives, and* $\mathrm{Hom}(M(\gamma), \mathbb{Q}(0)) = 0$.

PROOF. In order to show that $M(\gamma)$ is a non-trivial extension, it suffices to check that some realisation of $M(\gamma)$ is a non-trivial extension. Let us look at the Hodge realisation. The Hodge realisation of the exact sequence is a sequence of mixed Hodge modules whose fibre over $z \neq 0, 1$ is the sequence of mixed Hodge structures presented in the lower row of the following diagram.

$$\begin{array}{ccccccc}
 & & H^1(Y\cup f^{-1}(z), Y) & \xrightarrow{\cong} & H^1(\widetilde{Y}\cup f^{-1}(z)\cup E, \widetilde{Y}\cup E) & & \\
 & & \downarrow & & \downarrow & & \\
H^1(\mathbb{P}^1, \{0,\infty\}) & \longrightarrow & H^2(X, Y\cup f^{-1}(z)) & \xrightarrow{(*)} & H^2(\widetilde{X}, \widetilde{Y}\cup f^{-1}(z)) & \longrightarrow & H^2(\mathbb{P}^1, \{0,\infty\})
\end{array}$$

The vertical maps are morphisms induced by triples, and the top horizontal morphism is the induced by the blow-up map $\pi$ restricted to $\widetilde{Y}\cup f^{-1}(z)\cup E$. The morphism labelled $(*)$ is also induced by the blow-up map, hence the diagram commutes. The image of the morphism $(*)$ is the fibre over $z$ of the Hodge realisation of $M(\gamma)$. The top horizontal morphism is an isomorphism of Hodge structures, so the fibre over $z$ of the Hodge realisation of $M(\gamma)$ is the Hodge structure

$$H^1(\mathbb{G}_m, \{1, z^2\}) \cong H^1(f^{-1}(z), \{(1, z), (z, 1)\})$$

which is an extension of $\mathbb{Q}(-1)$ by $\mathbb{Q}$, non-split unless $z$ is a root of unity. $\square$

12.8.5 (Computation of rapid decay homology). — The topological picture is as follows: The topological space $X(\mathbb{C}) = \mathbb{C}^2$ has the homotopy type of a point. The subspace $Y(\mathbb{C})$ consists of four copies of the complex plane glued to a square, hence has the homotopy type of a circle. The set $f^{-1}(S_r) = \{(x, y) \in \mathbb{C}^2 \mid \mathrm{Re}(xy) \geqslant r\}$ is homeomorphic to $\mathbb{C}^* \times \mathbb{R} \times \mathbb{R}_{\geqslant 0}$, and is for $r > 1$ glued to $Y(\mathbb{C})$ in the adjacent lines $y = 1$ and $x = 1$. Here is the real picture.

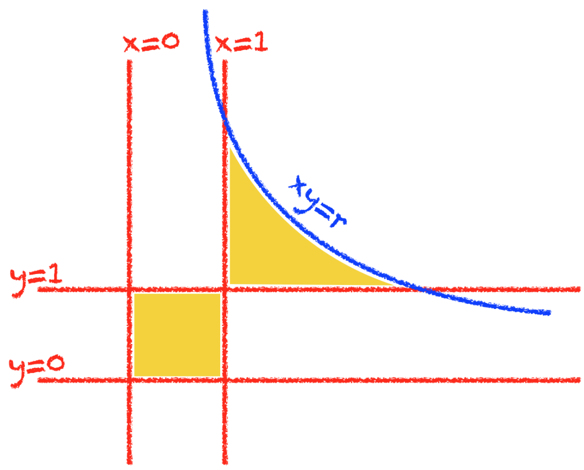



FIGURE 12.8.1

The space $Y(\mathbb{C}) \cup f^{-1}(S_r)$ has the homotopy type of a wedge of three circles. In the picture, two of these circles are visible as the boundaries of the yellow cells. The third circle is the circle in $\{(x, y) \in \mathbb{C}^2 \mid xy = r\}$, given by the simple loop $t \longmapsto (e^{2\pi it}, re^{-2\pi it})$. There is another subtlety which is invisible in the picture: The three sides of the cell attached to $f^{-1}(S_r)$ are paths from $(1, 1)$ to $(1, r)$ in the $x = 1$ plane, from $(1, 1)$ to $(r, 1)$ in the $y = 1$ plane, and from $(r, 1)$ to $(1, r)$ in the $xy = r$ plane. For the first two paths, any choice is homotopic to any other, but not so for the third since in the $xy = r$ plane a point is missing. Two choices for the boundary of the yellow triangle differ by a class in $H_1(f^{-1}(S_r), \mathbb{Q}) \simeq \mathbb{Q}$. A canonical choice is the cell which is contained in $\mathbb{R}^2 \subseteq \mathbb{C}^2$. The boundary morphism

$$\partial : H_2(X(\mathbb{C}), Y(\mathbb{C}) \cup f^{-1}(S_r)) \to H_1(Y(\mathbb{C}) \cup f^{-1}(S_r))$$

is an isomorphism. Therefore, $H_2(X(\mathbb{C}), Y(\mathbb{C}) \cup f^{-1}(S_r))$ has dimension 3, a basis being given by the two cells $\gamma_\square$ and $\gamma_\triangle$ in the picture, and a disk $\gamma_\circ$ filling the loop $t \longmapsto (e^{2\pi it}, re^{-2\pi it})$.

In order to compute a period matrix, we need to fix and keep track of orientations, and represent cycles in a way which is compatible with de Rham complexes. Let $\widetilde{Y}$ be the normalisation of $Y$, that is, the disjoint union of $Y_=$ and $Y_{||}$, where $Y_=$ stands for the two horizontal and $Y_{||}$ for the two vertical lines. Let $Z = Y_= \cap Y_{||} = \{(0,0),(0,1),(1,0),(1,1)\}$ be the four singular points of $Y$. The rapid decay homology of $[X, Y, f]$ is the homology of the total complex of the double complex

$$\begin{array}{ccccccccc}
0 & \longleftarrow & C_0(X,f) & \leftarrow & C_1(X,f) & \leftarrow & C_2(X,f) & \longleftarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longleftarrow & C_0(\widetilde{Y},f) & \leftarrow & C_1(\widetilde{Y},f) & \longleftarrow & \cdots & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longleftarrow & C_0(Z) & \longleftarrow & \cdots & & & &
\end{array}$$

where the horizontal complexes are singular chain complexes. The maps $C_n(\widetilde{Y}, f) \to C_n(X, f)$ are induced by the map $\widetilde{Y} \to X$, and $C_n(Z) \to C_n(\widetilde{Y}, f)$ is the map $i_{||} - i_=$, where $i_{||}$ and $i_=$ are induced by the inclusions $Z \subseteq Y_{||} \subseteq \widetilde{Y}$ and $Z \subseteq Y_= \subseteq \widetilde{Y}$. The cone of $C(Z) \to C(\widetilde{Y}, f)$ computes $C(Y, f)$ by Mayer–Vietoris.

12.8.6 (Computation of de Rham cohomology). — Let $\widetilde{Y}$ be the normalisation of $Y$ and let $Z \subseteq Y$ be the four singular points of $Y$. The de Rham complex associated with $[X, Y]$ and $f$ is the total complex of the double complex

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^0(X) & \xrightarrow{d_f} & \Omega^1(X) & \xrightarrow{d_f} & \Omega^2(X) & \longrightarrow & 0 \\
 & & {\scriptstyle\text{res}}\downarrow & & {\scriptstyle\text{res}}\downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^0(\widetilde{Y}) & \xrightarrow{d_f} & \Omega^1(\widetilde{Y}) & \longrightarrow & 0 & & \\
 & & {\scriptstyle\delta}\downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \Omega^0(Z) & \longrightarrow & 0 & & & &
\end{array}$$

which we describe now explicitely. Our conclusion will be that $H^2_{\mathrm{dR}}(X, Y, f)$ is of dimension 3, and a basis is represented by the triples $(\omega, 0, 0)$, $(0, 0, \delta_{00})$ and $(0, 0, \delta_{11})$ in $\Omega^2(X) \oplus \Omega^1(\widetilde{Y}) \oplus \Omega^0(Z)$ given by

$$\omega = dydx \qquad \text{and} \qquad \delta_{ij}(x,y) = \begin{cases} 1 & (x,y) = (i,j) \\ 0 & \text{otherwise} \end{cases} \tag{12.8.6.1}$$

The de Rham complex $(\Omega(X), d_f)$ is the complex $\mathbb{Q}[x,y] \to \mathbb{Q}[x,y]dx \oplus \mathbb{Q}[x,y]dy \to \mathbb{Q}[x,y]dxdy$ with differentials given by

$$d_f(g) = \Big(yg + \frac{\partial g}{\partial x}\Big)dx + \Big(xg + \frac{\partial g}{\partial y}\Big)dy \qquad \text{and} \qquad d_f(gdx + hdy) = \Big(-xg + yh + \frac{\partial g}{\partial y} - \frac{\partial h}{\partial x}\Big)dxdy$$

and its homology is concentrated in degree 2 of dimension 1, represented by the form $dxdy$. The variety $\widetilde{Y}$ is the union of four affine lines, say the spectrum of $\mathbb{Q}[z] \oplus \mathbb{Q}[z] \oplus \mathbb{Q}[z] \oplus \mathbb{Q}[z]$, where we arrange coordinates in such a way that a regular function $g = g(x, y)$ or a differential form $gdx + hdy$ on the plane $X$ restrict to

$$\left(\begin{array}{c|c|c} & g(z,1) & \\ \hline g(0,z) & & g(1,z) \\ \hline & g(z,0) & \end{array}\right) \qquad \text{and} \qquad \left(\begin{array}{c|c|c} & g(z,1)dz & \\ \hline h(0,z)dz & & h(1,z)dz \\ \hline & g(z,0)dz & \end{array}\right)$$

respectively. As the notation suggests, the entry on the top of these diagrams represents a function on the affine line $(y = 1)$ in $Y$, the entry on the left a function on the line $(x = 0)$, and so forth. The function $f(x, y) = xy$ restricts to

$$\operatorname{res}(f) = \left( \begin{array}{c|c|c} & z & \\ \hline 0 & & z \\ \hline & 0 & \end{array} \right)$$

hence the differential $d_f \colon \Omega^0(\widetilde{Y}) \to \Omega^1(\widetilde{Y})$ is given by

$$d_f \left( \begin{array}{c|c|c} & h_1 & \\ \hline g_0 & & g_1 \\ \hline & h_0 & \end{array} \right) = \left( \begin{array}{c|c|c} & (g_1 + g_1')dz & \\ \hline g_0' dz & & (h_1 + h_1')dz \\ \hline & h_0' dz & \end{array} \right) .$$

The homology of $\Omega(\widetilde{Y}), d_f)$ is concentrated in degree 0 of dimension 2, generated by the constant functions

$$\left( \begin{array}{c|c|c} & 0 & \\ \hline 1 & & 0 \\ \hline & 0 & \end{array} \right) \qquad \text{and} \qquad \left( \begin{array}{c|c|c} & 0 & \\ \hline 0 & & 0 \\ \hline & 1 & \end{array} \right) . \tag{12.8.6.2}$$

Elements of $\Omega^0(Z)$ are quadruples of rational numbers which we arrange in a matrix in the evident way. The map $\delta : \Omega^0(\widetilde{Y}) \to \Omega^0(Z)$ is given by

$$\delta \left( \begin{array}{c|c|c} & h_1 & \\ \hline g_0 & & g_1 \\ \hline & h_0 & \end{array} \right) = \begin{pmatrix} g_0(1) - h_1(0) & g_1(1) - h_1(1) \\ g_0(0) - h_0(0) & g_0(1) - h_1(0) \end{pmatrix}$$

A particular basis of $\Omega^0(Z)$ is given by the four elements

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

where the first two are the images under $\delta$ of the basis of $\ker(d_f)$ given in (12.8.6.2), and the second two are the elements $\delta_{00}$ and $\delta_{11}$ given in (12.8.6.1).

12.8.7 (The period matrix). — We now turn to the computation of the integrals of $\omega$, $\delta_{00}$ and $\delta_{11}$, each over the three topological cycles $\gamma_\square$, $\gamma_\triangle$ and $\gamma_\circ$. The following table (aka. period matrix) summarises the results:

| $\int$ | $\delta_{00}$ | $\delta_{11}$ | $\omega$ |
|---|---|---|---|
| $\gamma_\square - \gamma_\triangle$ | 1 | 0 | $\gamma$ |
| $\gamma_\triangle$ | 0 | $e^{-1}$ | $E_1(1)$ |
| $\gamma_\circ$ | 0 | 0 | $2\pi i$ |

The circle $\gamma_\circ$ does not meet the $(0, 0)$ or $(1, 1)$, and the triangle $\gamma_\triangle$ does not meet $(0, 0)$. This explains the zeroes below the diagonal in the period matrix. Calculating the integrals of $\delta_{00}$ and $\delta_{11}$ on $\gamma_\square$ and $\gamma_\triangle$ is a question of bookkeeping. The boundary of $\gamma_\square$ maps via the injective map $H_1(Y, f) \to H_0(Z, f)$ to $(0, 0) + (0, 1) + (1, 0) + (1, 1)$ in $H_0(Z)$. We find

$$\int_\square \delta_{00} = 1 \cdot e^{-0} = 1 \qquad \text{and} \qquad \int_\square \delta_{11} = 1 \cdot e^{-1} = e^{-1}$$

and similarly

$$\int_{\triangle} \delta_{11} = 1 \cdot e^{-1} =? \pm e^{-1}$$

[up to signs! before we discuss signs an orientation of the cycles has to be chosen!] It remains to evaluate the honest integrals. Setting $x = e^{2\pi i t}$ and $y = re^{-2\pi i t}$ we find

$$\int_{\circ} e^{-xy} dx dy = 2\pi i \int_0^1 \int_0^1 e^{-r} e^{2\pi i t} e^{-2\pi i t} dt dr = 2\pi i (1 - e^{-r})$$

and this quantity converges to $2\pi i$ as $r \to \infty$.

$$\int_{\triangle} e^{-xy} dx dy = \int_1^{\infty} \int_1^{\infty} e^{-xy} dx dy = \int_1^{\infty} y^{-1} e^{-y} dy = E_1(1)$$

REMARK 12.8.8. — The shape of the period matrix suggests that $H^2(X, Y, f)$ has a subobject or a quotient isomorphic to the motive associated to $E_1(1)$, as introduced in the previous section. Indeed, this is the case. Let $Z$ be the union of the lines $x = 1$ and $y = 1$. The exact sequence (4.2.4.2) for the pair of inclusions $Z \subseteq Y \subseteq X$ yields an exact sequence

$$0 \longrightarrow H^1(Y, Z, f|_Y) \longrightarrow H^2(X, Y, f) \longrightarrow H^2(X, Z, f) \longrightarrow 0$$

12.8.9 (Computation of Hodge realisation). — The perverse sheaf underlying the exponential Hodge realisation of $M$ has two singularities, $S = \{0, 1\}$.

CHAPTER 13

# Gamma motives and the abelianisation of the Galois group

## 13.1. The gamma motive

At the outset of these notes stands Lang's conjecture 1.3.4 about the transcendence degree of the field generated over $\overline{\mathbb{Q}}$ by the values of the gamma function at rational numbers with a fixed denominator. As we saw in Example 1.1.4 from the introduction, they all appear as periods of the following exponential motives over $\mathbb{Q}$:

$$M_n = H^1(\mathbb{A}^1_{\mathbb{Q}}, x^n).$$

Note that, if $n$ divides $m$, the map $x \longmapsto x^{m/n}$ induces an inclusion $M_n \subset M_m$. We call *gamma motive* the ind-object $\operatorname{colim}_n M_n$ of $\mathbf{M}^{exp}(k)$.

In this section, we shall compute the motivic Galois group of $M_n$ and explain the relation with the Serre torus of the cyclotomic field $\mathbb{Q}(\mu_n)$. From this we will deduce that Lang's conjecture is equivalent to the exponential period conjecture 8.2.3 for the motive $M_n$.

13.1.1 (Motives of Fermat hypersurfaces). — Given two integers $n, m \geqslant 2$, we consider the following variants of the Fermat hypersurface:

$$\begin{aligned} Y &= \{[x_0\colon \cdots : x_m] \in \mathbb{P}^m \mid x_0^n = x_1^n + \ldots + x_m^n\}, \\ X &= \{[x_1\colon \cdots : x_m] \in \mathbb{P}^{m-1} \mid x_1^n + \ldots + x_m^n = 0\}, \\ U &= \{(x_1, \ldots, x_m) \in \mathbb{A}^m \mid x_1^n + \cdots + x_m^n = 1\}. \end{aligned}$$

We shall regard them as varieties over the cyclotomic field $k = \mathbb{Q}(\mu_n)$ and write *e.g.* $X_n^{m-2}$ instead of $X$ when we want to emphasise the degree and the dimension. Observe that the map $[x_1\colon \ldots : x_m] \longmapsto [0\colon x_1\colon \ldots : x_m]$ induces a closed immersion $\iota\colon X \hookrightarrow Y$, whose open complement is $U$, under the identification $\mathbb{A}^m \simeq \mathbb{P}^{m+1} \setminus \{x_0 = 0\}$.

Following Anderson [**An86**, 10.2], we make the group $\Lambda = \bigoplus_{i=1}^m \mu_n$ act on $Y$ by

$$(\xi_1, \ldots, \xi_n) \cdot [x_0\colon x_1\colon \cdots : x_m] = [x_0\colon \xi_1 x_1\colon \cdots : \xi_m x_m].$$

This action stabilizes both $X$ and $U$. We identify the character group of $\Lambda(\mathbb{C})$ with $(\mathbb{Z}/n)^m$ by associating to an element $\underline{a} = (a_1, \ldots, a_m) \in (\mathbb{Z}/n)^m$ the character $(\xi_1, \ldots, \xi_m) \longmapsto \prod_{i=1}^m \xi_i^{a_i}$. Set

$$\Psi = \{\underline{a} = (a_1, \ldots, a_m) \in (\mathbb{Z}/n)^m \mid a_i \neq 0,\ a_1 + \cdots + a_m = 0\}.$$

Then there is a decomposition

$$H^{m-2}(X) = \iota^* H^*(\mathbb{P}^{m-1}) \oplus \bigoplus_{a \in \Psi} H_a^{m-2}, \tag{13.1.1.1}$$

It follows that the primitive cohomology $H^{m-2}_{\mathrm{prim}}(X)$ is cut out in $H^{m-2}(X)$ by the projector

$$\theta_{\mathrm{prim}} = \frac{1}{m^n} \sum_{\lambda \in \Lambda(\mathbb{C})} \sum_{a \in \Psi} (a, \lambda)\lambda.$$

The map $[x_1 \colon \ldots \colon x_m] \longmapsto [0 \colon x_1 \colon \ldots \colon x_m]$ induces a closed immersion $\iota \colon X \hookrightarrow Y$, whose open complement is $U$, under the identification $\mathbb{A}^m \simeq \mathbb{P}^{m+1} \setminus \{x_0 = 0\}$. Noting that $X$ is a smooth divisor on $Y$, the Gysin exact sequence of motives reads:

$$\cdots \longrightarrow H^i(Y) \longrightarrow H^i(U) \longrightarrow H^{i-1}(X)(-1) \longrightarrow H^{i+1}(Y) \longrightarrow \cdots \tag{13.1.1.2}$$

Moreover, (13.1.1.2) is $\Lambda$-equivariant, so we can replace each term with its image under the projector $\theta_{\mathrm{prim}}$ and still get an exact sequence. Since $\theta_{\mathrm{prim}}$ annhilates the cohomology of $Y$, it follows that:

$$H^{m-1}_{\mathrm{prim}}(U) \xrightarrow{\sim} H^{m-2}_{\mathrm{prim}}(X)(-1).$$

We introduce the differential form

$$\Omega = \sum_{\ell=1}^{m} (-1)^{\ell} x_\ell dx_1 \wedge \cdots \wedge \widehat{dx_\ell} \wedge \cdots \wedge dx_m.$$

13.1.2 (Tensor powers of the gamma motive). — We now have all the ingredients to prove that, for each integer $m \geqslant 2$, the tensor power $M_n^{\otimes m}$ contains a submotive isomorphic to $H^{m-2}_{\mathrm{prim}}(X)(-1)$.

Proposition 13.1.3. — *There is an isomorphism of exponential motives*

$$(M_n^{\otimes m})^{\mu_n} \xrightarrow{\sim} H^{m-2}_{\mathrm{prim}}(X_n^{m-2})(-1). \tag{13.1.3.1}$$

Proof. The proof is an elaboration on the ideas that were already used in Lemma 12.2.1. We first recall that, by the Künneth formula,

$$M_n^{\otimes m} = H^m(\mathbb{A}^m, x_1^n + \cdots + x_m^n).$$

Consider the morphism $h \colon U \times \mathbb{A}^1 \to \mathbb{A}^m$ given by

$$h((x_1, \ldots, x_m), r) = (rx_1, \ldots, rx_m).$$

Since $h$ sends the closed subvariety $U \times \{0\} \subseteq U \times \mathbb{A}^1$ to the origin $O \in \mathbb{A}^1$ and commutes with the functions $0 \boxplus r^n$ on $U \times \mathbb{A}^1$ and $x_1^n + \ldots + x_m^n$ on $\mathbb{A}^m$, it induces a morphism of motives

$$H^m(\mathbb{A}^m, O, x_1^n + \ldots + x_n^m) \longrightarrow H^m(U \times \mathbb{A}^1, U \times \{0\}, 0 \boxplus r^n).$$

The source is isomorphic to $H^m(\mathbb{A}^m, x_1^n + \ldots + x_n^m)$ by the long exact sequence (4.2.4.2) and the target decomposes as a tensor product according to the Künneth formula, so we get a map:

$$M_n^{\otimes m} \longrightarrow H^{m-1}(U) \otimes H^1(\mathbb{A}^1, \{0\}, r^n). \tag{13.1.3.2}$$

We need to show that the morphism

$$(M_n^{\otimes m})^{\mu_n} \longrightarrow H^{m-1}(U) \otimes H^1(\mathbb{A}^1, r^n) \tag{13.1.3.3}$$

obtained from (13.1.3.2) by restricting to the submotive $(M_n^{\otimes m})^{\mu_n} \subseteq M_n^{\otimes m}$ and composing with the projection $H^1(\mathbb{A}^1, \{0\}, r^n) \to H^1(\mathbb{A}^1, r^n)$ is identically zero. This will yield a morphism

$$(M_n^{\otimes m})^{\mu_n} \longrightarrow H^{m-1}(U) \tag{13.1.3.4}$$

and the proof will show as well that (13.1.3.4) is injective with image $H^{m-1}_{\mathrm{prim}}(U)$.

To carry out this program we look at the de Rham realisation. A basis of $H^m_{\mathrm{dR}}(\mathbb{A}^m, x_1^n+\ldots+x_m^n)$ is given by the differentials

$$\omega_{\underline{j}} = x_1^{j_1-1}\cdots x_m^{j_m-1}dx_1\cdots dx_m, \qquad \underline{j} = (j_1,\ldots,j_m) \in \{1,\ldots,n-1\}^m,$$

which are $\mu_n$-invariant if and only if $n$ divides $|\underline{j}| = j_1+\ldots+j_m$. By a straightforward computation, the morphism $h$ sends this basis to

$$\begin{aligned} h^*\omega_{\underline{j}} &= \sum_{\ell=1}^{m}(-1)^{m-\ell}x_1^{j_1-1}\cdots x_\ell^{j_\ell}\cdots x_m^{j_m-1}dx_1\cdots \widehat{dx_\ell}\cdots dx_m \otimes r^{|\underline{j}|-1}dr \\ &= (-1)^m x_1^{j_1-1}\cdots x_m^{j_m-1}\Omega \otimes r^{|j|-1}dr. \end{aligned}$$

Let us now assume that $n$ divides $|\underline{j}|$. By induction, the relation

$$r^{an-1}dr - \tfrac{1}{(a-1)}r^{(a-1)n-1}dr = d_{r^n}(-\tfrac{1}{n}r^{(a-1)n})$$

implies that the differentials $r^{an-1}dr$ and $\frac{1}{(a-1)!}r^{n-1}dr$ are cohomologous for all integers $a \geqslant 1$. Taking into account that $r^{n-1}dr$ spans the kernel of the projection $H^1_{\mathrm{dR}}(\mathbb{A}^1, \{0\}, r^n) \to H^1_{\mathrm{dR}}(\mathbb{A}^1, r^n)$, it follows that (13.1.3.3) realises to the zero map in de Rham cohomology, hence it is itself zero. The argument also shows that the resulting morphism

$$(R_{\mathrm{dR}}(M_n)^{\otimes m})^{\mu_n} \longrightarrow H^{m-1}_{\mathrm{dR}}(U)$$

sends the basis $[\omega_{\underline{j}}]$, where $\underline{j}$ runs through the indices such that $n$ divides $|\underline{j}|$, to

$$\frac{(-1)^m}{(|j|-1)!}[x_1^{j_1-1}\cdots x_m^{j_m-1}\Omega].$$

To conclude, it suffices to show that these classes form a basis of $H^{m-1}_{\mathrm{dR,prim}}(U)$. □

Remark 13.1.4. — Let us analyse the content of the proposition for $m = 2$. Set $\zeta = e^{\frac{\pi i}{n}}$. The variety $X_n^0 \subseteq \mathbb{P}^1$ is the finite set of points $P_r = [1\colon \zeta^{2r-1}]$ for $r \in \mathbb{Z}/n$. The group $\Lambda = \mu_n^2$ permutes these points as follows:

$$(e^{\frac{2\pi i a_1}{n}}, e^{\frac{2\pi i a_2}{n}}) \cdot P_r = P_{a_2-a_1+r}.$$

In particular, if $a_1+a_2 \equiv 0$, then $P_r$ is sent to $P_{r-2a_1}$. Now recall that the gamma function satisfies

$$\Gamma(\tfrac{j}{n})\Gamma(1-\frac{j}{n}) = \frac{\pi}{\sin(\frac{\pi j}{n})} = \frac{2\pi i}{\zeta^j+\zeta^{n-j}}.$$

Remark 13.1.5. — Here is how the fact that $(M_n^{\otimes m})^{\mu_n}$ is isomorphic to a usual motive is reflected at the level of the irregular Hodge filtration. A basis of $R_{\mathrm{dR}}(M_n^{\otimes m})$ is given by the elements

$$x_1^{j_1-1}dx_1 \otimes \cdots \otimes x_m^{j_m-1}dx_m, \quad 1 \leqslant j_i \leqslant n-1, \tag{13.1.5.1}$$

which are pure of Hodge type $(\frac{j_1+\ldots+j_m}{n}, m-\frac{j_1+\ldots+j_m}{n})$. This type is integral if and only if $j_1+\ldots+j_m$ is a multiple of $n$. Since $\xi$ acts on (13.1.5.1) by multiplication by $\xi^{j_1+\cdots+j_m}$, the $\mu_n$-invariant differentials are exactly those having integral Hodge type.

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# List of symbols

| | |
|---|---|
| $\mathcal{L}_\psi$ | the Artin-Schreier sheaf associated to an additive character, page 177 |
| $\mathrm{Q}^{\exp}(k)$ | the quiver of exponential relative varieties over $k$, page 104 |
| $\langle Q, \rho\rangle$ | the linear hull of a quiver representation $\rho\colon Q \to \mathbf{Vec}_{\mathbb{Q}}$, page 97 |
| $G_M$ | the Galois group of an exponential motive $M$, page 123 |
| $\mathrm{G}^{\exp}(k)$ | the exponential motivic Galois group, page 123 |
| $R\Psi_p$ | the nearby cycles functor from perverse sheaves on $\mathbb{A}^1_{\mathbb{Q}}$ to perverse sheaves on $\mathbb{A}^1_{\mathbb{F}_p}$, page 177 |
| $\mathbf{PS}(k)$ | the category of period structures over $k$, page 169 |
| $\mathbf{Perv}$ | the category of $\mathbb{Q}$-perverse sheaves on $\mathbb{A}^1(\mathbb{C})$, page 32 |
| $\mathbf{Perv}_0$ | the category of $\mathbb{Q}$-perverse sheaves on $\mathbb{A}^1(\mathbb{C})$ with no global cohomology, page 32 |
| $\Psi_\infty$ | the nearby fibre at infinity functor on the category $\mathbf{Perv}_0$, page 33 |

# Index