

INHOMOGENEOUS MINIMA OF MIXED SIGNATURE LATTICES

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ABSTRACT. We establish an explicit upper bound for the Euclidean minimum of a number field which depends, in a precise manner, only on its discriminant and the number of real and complex embeddings. Such bounds were shown to exist by Davenport and Swinnerton-Dyer ([Dav52, Dav53, DS55]). In the case of totally real fields, an optimal bound was conjectured by Minkowski and it is proved for fields of small degree. In this note we develop methods of McMullen ([McM05]) in the case of mixed signature in order to get explicit bounds for the Euclidean minimum.

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1. INTRODUCTION

Let K be a number field of degree n , let \mathcal{O}_K be its ring of integers and d_K be the absolute value of its discriminant. Let $N : K \rightarrow \mathbb{Q}$ be the absolute value of the norm map. The number field K is said to be *Euclidean with respect to the norm* if for every $a, b \in \mathcal{O}_K$ with $b \neq 0$ there exist $c, d \in \mathcal{O}_K$ such that $a = bc + d$ and $N(d) < N(b)$. Equivalently, the number field K is Euclidean with respect to the norm if for every $x \in K$ there exists $c \in \mathcal{O}_K$ such that $N(x - c) < 1$. This suggests to look at the real number

$$(1) \quad M(K) := \sup_{x \in K} \inf_{c \in \mathcal{O}_K} N(x - c),$$

called the *Euclidean minimum* of K . If K is totally real, then Minkowski's conjecture states that the inequality

$$(2) \quad M(K) \leq 2^{-n} \cdot \sqrt{d_K}$$

holds. This conjecture is known for $n \leq 9$ (see §2.3 for details). It is natural to look for bounds similar to (2) for number fields of mixed signature. Such bounds were obtained by Clarke [Cla51]

and Davenport [Dav52]. The latter proved that for every pair of nonnegative integers (r, s) there exists a constant $C_{r,s}$ such that

$$(3) \quad M(K) \leq 2^{\frac{-sn}{r+s}} \cdot C_{r,s} \cdot d_K^{\frac{n}{2(r+s)}}$$

holds for every number field K of signature (r, s) . Although an explicit constant $C_{r,s}$ can be deduced from Davenport's proof, it is too large to be useful. The aim of this paper is to develop methods of McMullen (cf. [McM05]) to obtain a better constant. A weakened but easy to read form of our main result (Theorem 5.3) is the following:

Theorem. *Let K be a number field of signature (r, s) and degree $n = r + 2s \geq 4$, and let d_K be the absolute value of the discriminant of K . The following inequality holds.*

$$M(K) \leq 2^{\frac{-sn}{r+s}} \cdot \left(\frac{1}{2}\sqrt{n}\right)^n \cdot d_K^{\frac{n}{2(r+s)}}.$$

In other words, the inequality (3) holds with the constant $C_{r,s} = 2^{-n} \cdot n^{\frac{n}{2}}$, which is still very large. In §2 we compare our result to known estimates for totally real and totally imaginary fields, and formulate some questions as to which bounds one might hope for.

Convention: Throughout the paper, all vector spaces are understood to be finite dimensional real vector spaces. A *scalar product* on a vector space V is a symmetric positive definite bilinear form $V \times V \rightarrow \mathbb{R}$, and by a *lattice* in V we understand a cocompact discrete subgroup of V .

2. INHOMOGENEOUS MINIMA OF LATTICES OF MIXED SIGNATURE

We explain in this section what we mean by lattices of mixed signature, recall the definition and some properties of inhomogeneous minima, and give some motivation for studying these objects.

– **2.1.** Let r and s be nonnegative integers and set $n = r + 2s$. Let $V := \mathbb{R}^r \oplus \mathbb{C}^s$, which is an n -dimensional vector space over \mathbb{R} . We equip V with the scalar product $\langle -, - \rangle$ given by

$$(4) \quad \langle (v_1, \dots, v_{r+s}), (v'_1, \dots, v'_{r+s}) \rangle = \sum_{i=1}^r v_i v'_i + \sum_{i=r+1}^{r+s} \operatorname{Re}(v_i \bar{v}'_i)$$

and we call the function $N : V \rightarrow \mathbb{R}$ given by

$$(5) \quad N(v_1, \dots, v_{r+s}) = |v_1 \cdots v_r \cdot v_{r+1}^2 \cdots v_{r+s}^2|$$

the *norm*, although it is not a norm in the usual sense.

By a *lattice of signature (r, s)* we mean a lattice in this particular vector space V . We denote by $\det(\Lambda)$ the volume of V/Λ with respect to the volume form obtained from the scalar product (4). We call the real numbers

$$m(\Lambda) := \inf_{\lambda \in \Lambda \setminus \{0\}} N(\lambda) \quad \text{and} \quad M(\Lambda) := \sup_{v \in V} \inf_{\lambda \in \Lambda} N(v - \lambda)$$

the *homogeneous minimum*, respectively the *inhomogeneous minimum*, of Λ .

– **2.2.** Our motivation for studying inhomogeneous minima of lattices of mixed signature is the classical geometry of numbers. Let K be a number field of degree n and signature (r, s) , so that $n = r + 2s$. Let d_K be the absolute value of the discriminant of K . Choosing an ordering of the r real and of s non-conjugated complex embeddings of K we obtain a \mathbb{Q} -linear embedding $K \rightarrow V$ with dense image. The image of the ring of integers \mathcal{O}_K of K in V is a lattice of volume $2^{-s} \cdot \sqrt{d_K}$ and the norm map $N : V \rightarrow \mathbb{R}$ given in (5) continuously extends the absolute value of the usual norm map $N : K \rightarrow \mathbb{Q}$, hence the name. In this context, $m(K)$ and $M(K)$ denote the homogeneous and the inhomogeneous minimum of \mathcal{O}_K as a lattice in V . These quantities are independent of the ordering of the real and complex embeddings of K . Cerri proves in [Cer06] that $M(K)$ is equal to the Euclidean minimum of the number field K as given in (1), which is a priori a smaller quantity.

– **2.3.** *Minkowski's conjecture* on inhomogeneous minima of products of real linear forms (see for instance [GL87]) states that if $s = 0$, the inequality

$$M(\Lambda) \leq 2^{-n} \cdot \det(\Lambda)$$

holds for every lattice Λ in V . The conjecture is proved for n up to 9 ([Min00, Rem23, Dys48, Sku72, McM05, HRS09, HRS11, KR14]). In terms of number fields, Minkowski's conjecture implies that

$$M(K) \leq 2^{-n} \cdot \sqrt{d_K}$$

holds for every totally real number field K . This is proved also for particular totally real fields of degree $n > 9$, see for example [BN05, Bay06, BS06, BM13]. For totally real number fields of any degree n , the inequality

$$(6) \quad M(K) \leq (\sqrt{2})^{-n} \cdot \sqrt{d_K}$$

holds by a theorem of Chebotarev, see for instance [Dav46]. There are improvements of this estimate, yet, to our best knowledge, it is at present not known whether $M(K) \leq c^{-n} \cdot \sqrt{d_K}$ holds for some real number $c > \sqrt{2}$.

– **2.4.** Improving earlier results of Clarke ([Cla51]), Davenport shows in [Dav52] that for every pair (r, s) there exists a real number $C_{r,s} \geq 0$ such that the inequality (with the convention $0^0 = 1$)

$$(7) \quad m(\Lambda)^{\frac{s}{r+s}} \cdot M(\Lambda) \leq C_{r,s} \cdot \det(\Lambda)^{\frac{n}{r+s}}$$

holds for all lattices of signature (r, s) . Notice that if we scale Λ by a real number $t > 0$, both sides of the inequality change by the factor $t^{n^2/(r+s)}$. Also notice that the inequality is trivial if $m(\Lambda) = 0$, unless $s = 0$. In terms of number fields, where we have $m(\Lambda) = 1$, Davenport's result states that

$$(8) \quad M(K) \leq 2^{-\frac{sn}{r+s}} \cdot C_{r,s} \cdot d_K^{\frac{n}{2(r+s)}}.$$

holds for every number field K of signature (r, s) . In the case $s = 0$ one hopes that the inequality (7) holds with the constant $C_{r,0} = 2^{-r}$, according to Minkowski's conjecture, and one knows (7) to hold for $C_{r,0} = (\sqrt{2})^{-r}$, according to Chebotarev's theorem. In [Bay06] it is proved that

$$(9) \quad M(K) \leq 2^{-n} \cdot d_K$$

holds for any number field K of degree n . In other words, for $r = 0$ the inequality (8) holds with the constant $C_{0,s} = 1$.

– **2.5.** We wonder what an analogue of Minkowski’s conjecture for number fields or lattices of mixed signature should look like. In an updated version of [Lem95] of 2004, Lemmermeyer states that “similar results [to Minkowski’s conjecture] (not even a conjecture) for fields with mixed signature are not known except for a theorem of Swinnerton-Dyer ([Swi54]) concerning cubic fields”. The result in question states that the inequality

$$(10) \quad M(K) \leq \frac{1}{16\sqrt[3]{2}} \cdot d_K^{\frac{2}{3}}$$

holds when K is a complex cubic field. Also, the Euclidean minimum of any complex quadratic field K is known, and we have that $M(K) \leq \frac{1}{8} \cdot d_K$.

– **2.6.** If we agree that a mixed signature analogue of Minkowski’s conjecture is an estimate of the form (7), then the quantity we search to determine is the real number

$$c_{r,s} := \sup \left\{ m(\Lambda)^{\frac{s}{r+s}} \cdot M(\Lambda) \cdot \det(\Lambda)^{\frac{-n}{r+s}} \mid \Lambda \text{ is a lattice of signature } (r, s) \right\}$$

which exists by Davenport’s result. Minkowski’s conjecture states $c_{r,0} = 2^{-r}$, but moreover it is part of the conjecture that the supremum is attained precisely for those lattices $\Lambda \subseteq \mathbb{R}^r$ which are sums of r lattices $\Lambda_i = a_i \mathbb{Z} \subseteq \mathbb{R}$. For instance, the standard lattice $\mathbb{Z}^r \subseteq \mathbb{R}^r$ satisfies $M(\mathbb{Z}^r) = 2^{-r}$. In the case of mixed signature (r, s) , lattices which are sums of r lattices $a_i \mathbb{Z} \subseteq \mathbb{R}$ and s lattices $x_i \mathbb{Z} + y_i \mathbb{Z} \subseteq \mathbb{C}$ are not helpful in guessing $c_{r,s}$, because their homogeneous minima are zero. Indeed, the equality

$$c_{r,s} = \sup \left\{ M(\Lambda) \cdot \det(\Lambda)^{\frac{-n}{r+s}} \mid \Lambda \text{ is a lattice of signature } (r, s) \text{ and } m(\Lambda) = 1 \right\}$$

holds for $s > 0$. It is not clear whether this is true as well when $s = 0$, except in the case $(r, s) = (2, 0)$ because there are real quadratic number fields whose euclidean minimum is arbitrarily close to Minkowski’s bound. Also, inhomogeneous minima of complex quadratic fields are known, and one can show that $c_{0,1} = \frac{1}{2}$.

Question 2.7. Are there positive real numbers a, b such that $c_{r,s} \leq a^r b^s$ holds for all (r, s) , that is, such that the inequality

$$M(\Lambda) \leq a^r b^s \cdot \det(\Lambda)^{\frac{n}{r+s}}$$

holds for every lattice Λ of signature (r, s) and inhomogeneous minimum $m(\Lambda) = 1$? Can one choose $a = (\sqrt{2})^{-1}$ and $b = 1$ as suggested by Chebotarev’s and estimates (6) and (9)? Could one even choose $a = b = \frac{1}{2}$?

– **2.8.** The inequality (10) does not fit in the discussion of §2.6, because the exponent of the discriminant is not what we were asking for. This is not an accident that just happens in the case of cubic fields, indeed, Davenport and Swinnerton-Dyer have shown (cf. Theorem 1 in [DS55], together with [Dav53, Swi54]) that there exists a real number $B_{r,s} > 0$ such that

$$M(\Lambda) \leq B_{r,s} \cdot \det(\Lambda)^{\max\left(\frac{n-1}{r+s}, \frac{n-s}{(r+s)-s/2}\right)}$$

holds for every lattice $\Lambda \subseteq \mathbb{R}^r \oplus \mathbb{C}^s$ satisfying $m(\Lambda) = 1$.

Question 2.9. Let $A_{r,s} \subseteq \mathbb{R}$ be the set of those real numbers $\alpha \in \mathbb{R}$ with the property that there exists $C \in \mathbb{R}$ such that $M(\Lambda) \leq C \det(\Lambda)^\alpha$ holds for every lattice Λ of signature (r, s) and $m(\Lambda) = 1$. This set is either a closed or an open half line

$$A_{r,s} = (\alpha_{r,s}, \infty) \quad \text{or} \quad A_{r,s} = [\alpha_{r,s}, \infty)$$

and we can define constants $c_{r,s}(\alpha)$ analogous to those in 2.6 and ask questions like those in 2.7 for any exponent $\alpha \in A_{r,s}$. As proven by Davenport and Swinnerton-Dyer in *loc.cit.* we have

$$\alpha_{r,s} \leq \max\left(\frac{n-1}{r+s}, \frac{n-s}{(r+s)-s/2}\right)$$

and this inequality is an equality if $r = 0$ (so $\alpha_{0,s} = 2$) and if $r = s = 1$ (so $\alpha_{1,1} = \frac{4}{3}$). Davenport and Swinnerton-Dyer suggest also that we might have equality whenever $s = 1$, and also that the inequality is strict in other cases. What is the number $\alpha_{r,s}$? Do we have $\alpha_{r,s} \in A_{r,s}$?

3. RESULTS ON SUCCESSIVE MINIMA

We fix for this section a pair of nonnegative integers (r, s) , and denote by $V = \mathbb{R}^r \oplus \mathbb{C}^s$ the vector space of dimension $n = r + 2s$ introduced in 2.1, equipped with its scalar product (4) and norm map (5). We set $\|v\| := \sqrt{\langle v, v \rangle}$. We will recall the definition and some results concerning the successive minima of a lattice.

Definition 3.1. The *successive minima* of a lattice Λ in V are the real numbers $\mu_1(\Lambda), \dots, \mu_n(\Lambda)$ defined in the following way: for $m \in \{1, \dots, n\}$, $\mu_m(\Lambda)$ is the infimum of the real numbers r such that there exist m independent vectors $\lambda_1, \dots, \lambda_m$ in $\Lambda \setminus \{0\}$ with $\|\lambda_i\| \leq r$ for $i \in \{1, \dots, m\}$. Vectors $\lambda \in \Lambda$ with $\|\lambda\| = \mu_1(\Lambda)$ are called *shortest (nonzero) vectors*.

Definition 3.2. The *Hermite constant* for dimension n is $\gamma_n := \sup_{\Lambda} \mu_1(\Lambda)^2 \cdot \det(\Lambda)^{-2/n}$, where Λ runs over all lattices in V .

Lemma 3.3 (Minkowski). *For every lattice Λ in V and $1 \leq t \leq n$ we have*

$$\mu_1(\Lambda) \cdots \mu_t(\Lambda) \leq \gamma_n^{t/2} \cdot \det(\Lambda)^{t/n},$$

where γ_n is the Hermite constant.

Proof. This is Theorem 2.6.8 of [Mar03]. □

Lemma 3.4. *For every lattice Λ in V the following inequality holds:*

$$m(\Lambda) \leq \left(\frac{\sqrt{2}}{\sqrt{n}} \cdot \mu_1(\Lambda) \right)^n.$$

Proof. For every element $v = (v_1, \dots, v_{r+s})$ of V we have

$$\sum_{i=1}^r v_i^2 + \sum_{i=r+1}^{r+s} |v_i|^2 \leq 2\|v\|^2 = 2 \sum_{i=1}^r v_i^2 + \sum_{i=r+1}^{r+s} |v_i|^2$$

hence

$$(11) \quad N(v)^{\frac{1}{n}} \leq \frac{\sqrt{2}}{\sqrt{n}} \cdot \|v\|$$

by the inequality between arithmetic and geometric means. The desired inequality follows from

$$m(\Lambda)^{\frac{1}{n}} = \inf_{\lambda \neq 0} N(\lambda)^{\frac{1}{n}} \leq \frac{\sqrt{2}}{\sqrt{n}} \cdot \inf_{\lambda \neq 0} \|\lambda\| = \frac{\sqrt{2}}{\sqrt{n}} \cdot \mu_1(\Lambda)$$

where the infima are taken over all non-zero elements λ of Λ . □

Lemma 3.5. *The following inequality holds for every lattice Λ in V :*

$$M(\Lambda) \leq \left(\frac{1}{\sqrt{2}} \cdot \mu_n(\Lambda) \right)^n.$$

Proof. Let $v \in V$. By definition of the successive minima, there exist linearly independent elements $\lambda_1, \dots, \lambda_n$ of Λ satisfying $\|\lambda_i\| \leq \mu_n(\Lambda)$ for all $i \in \{1, \dots, n\}$. Choose an orthonormal basis e_1, \dots, e_n of V such that each λ_i is written as

$$\lambda_i = \sum_{j=1}^i a_{ij} e_j,$$

in other words, the matrix (a_{ij}) is upper triangular. Write $v = b_1 e_1 + \dots + b_n e_n$ and successively choose integers k_1, \dots, k_n such that the coefficients b'_j in

$$v - (k_1 \lambda_1 + \dots + k_n \lambda_n) = \sum_{j=1}^n b'_j e_j$$

satisfy $|b'_j| \leq \frac{1}{2} |a_{jj}|$. Because of $\|\lambda_i\| \leq \mu_n(\Lambda)$ we have $|a_{ij}| \leq \mu_n(\Lambda)$ and hence $|b'_j| \leq \frac{1}{2} \mu_n(\Lambda)$. Setting $\lambda = k_1 \lambda_1 + \dots + k_n \lambda_n$, we obtain the inequality

$$\|v - \lambda\|^2 = \sum_{j=1}^n |b'_j|^2 \leq \frac{n}{4} \cdot \mu_n^2(\Lambda)$$

or equivalently $\|v - \lambda\| \leq \frac{\sqrt{n}}{2} \cdot \mu_n(\Lambda)$. From the inequality between arithmetic and geometric means, as in (11), we obtain

$$N(v - \lambda) \leq \left(\frac{\sqrt{2}}{\sqrt{n}} \cdot \|v - \lambda\| \right)^n \leq \left(\frac{1}{\sqrt{2}} \cdot \mu_n(\Lambda) \right)^n,$$

and since v was arbitrary, the desired inequality follows. □

4. McMULLEN'S TOPOLOGICAL METHODS

A lattice $\Lambda \subseteq \mathbb{R}^n$ is said to be *well rounded* if all of its successive minima are equal. A conjecture of Woods ([Woo72]) predicts that the *covering radius* $\rho(\Lambda)$ of a well rounded lattice Λ satisfies the inequality

$$\rho(\Lambda) := \sup_{v \in \mathbb{R}^n} \inf_{\lambda \in \Lambda} \|v - \lambda\| \leq \frac{\sqrt{n}}{2} \cdot \det(\Lambda)^{\frac{1}{n}}.$$

Covering radii and inhomogeneous minima are linked by the inequality between arithmetic and geometric means

$$N(v)^{\frac{1}{n}} \leq \frac{\|v\|}{\sqrt{n}},$$

so that, to prove Minkowski's conjecture for $\Lambda \subseteq \mathbb{R}^n$ it is sufficient to show that there exists a well rounded lattice Λ' with the same inhomogeneous minimum as Λ , and to prove that Woods' conjecture holds for Λ' .

The set of lattices in \mathbb{R}^n can be identified with the quotient $\mathrm{GL}_n(\mathbb{R})/\mathrm{GL}_n(\mathbb{Z})$ and in particular it has the structure of a differentiable real manifold of dimension n^2 with a differentiable transitive left action by $\mathrm{GL}_n(\mathbb{R})$. McMullen proved in [McM05] that if the closure of the orbit of a lattice $\Lambda \subseteq \mathbb{R}^n$ under the group of diagonal matrices with positive entries and determinant 1 is compact, then some lattice in that closure is well rounded.

In the case where the orbit of $\Lambda \subseteq \mathbb{R}^n$ is already compact, McMullen's result follows with some effort from the following theorem (Theorem 1.6 of *loc.cit.*).

Theorem 4.1. *Let T be a real torus and let U_1, \dots, U_m be open subsets of T which cover T . Suppose that the inequality*

$$\mathrm{rank}(H_1(W, \mathbb{Z}) \rightarrow H_1(T, \mathbb{Z})) < p$$

holds for every $p \in \{1, \dots, m\}$ and every connected component W of U_p . Then m is strictly larger than the dimension of T .

– **4.2.** Let $V = \mathbb{R}^r \oplus \mathbb{C}^s$ be the vector space introduced in 2.1, equipped with its scalar product and the norm map. We write G for the subgroup of $\mathrm{SL}_{r+s}(\mathbb{R})$ consisting of the diagonal matrices with positive entries. The group G is a real Lie group, isomorphic to \mathbb{R}^{r+s-1} . It acts by \mathbb{R} -linear transformations on V and preserves the norm. Hence, G acts on the space of lattices in V and preserves homogeneous and inhomogeneous minima.

Theorem 4.3. *Let $\Lambda \subseteq V$ be a lattice such that $G\Lambda$ is compact. Then there exists $g \in G$ such that the equalities $\mu_{s+1}(g\Lambda) = \dots = \mu_n(g\Lambda)$ hold.*

Remark 4.4. For $s = 0$, Theorem 4.3 is McMullen's Theorem 4.1 in [McM05]. With a few adaptations, McMullen's method works also in our superficially more general setup. For the convenience of the reader, we check this in detail. Notice that we cannot expect to obtain more equalities between successive minima than stated. This is clear from the case $r = 0$. We could alternatively demand for any $0 \leq k \leq s$ the equalities $\mu_{1+k}(g\Lambda) = \dots = \mu_{r+s+k}(g\Lambda)$ to hold.

– **4.5.** We fix for the rest of this section a lattice $\Lambda \subseteq V := \mathbb{R}^r \oplus \mathbb{C}^s$ such that $G\Lambda$ is compact. To say that $G\Lambda$ is compact is to say that the stabiliser

$$G_\Lambda := \{g \in G \mid g\Lambda = \Lambda\}$$

of $\Lambda \subseteq V$ is cocompact in G . Yet in other words, since G_Λ is discrete in $G \simeq \mathbb{R}^{r+s-1}$, the quotient $T := G/G_\Lambda$ has to be a real torus of dimension $r + s - 1$. For every $g \in G$, we define:

$$\begin{aligned} D(g) &:= \{\lambda \in \Lambda \mid \|g\lambda\| < \mu_n(g\Lambda)\} \\ M(g) &:= \text{the real subspace of } V \text{ generated by } D(g) \end{aligned}$$

The set $D(g)$ is a finite and nonempty subset of Λ , and $M(g) \subseteq V$ is a proper rational subspace, by which we mean that $M(g)$ is not equal to V , and that $M(g)$ is generated by lattice vectors. Notice that $\dim M(g)$ is the smallest integer $p \geq 0$ for which $\mu_{p+1}(g\Lambda) = \dots = \mu_n(g\Lambda)$ holds. Thus, in order to prove Theorem 4.3, we have to show that there exists a $g \in G$ with $\dim M(g) \leq s$.

– **4.6.** For every integer $1 \leq p \leq n$, let us define $G(p)$ as the set of those $g \in G$ with $\dim M(g) = n - p$. The subsets $G(p)$ of G form a partition of G , and these sets are stable under the translation action of G_Λ . Indeed, for every $g \in G$ and $h \in G_\Lambda$ we have

$$D(gh) = \{\lambda \mid \|gh\lambda\| < \mu_n(g\Lambda)\} = \{h^{-1}\lambda \mid \|g\lambda\| < \mu_n(g\Lambda)\} = h^{-1}D(g)$$

hence $M(gh) = h^{-1}M(g)$ and $\dim M(gh) = \dim M(g)$. Therefore, as p ranges over $1, 2, \dots, n$, the sets

$$T(p) := G(p)/G_\Lambda = \{gG_\Lambda \mid \dim M(g) = n - p\}$$

form a partition of the torus $T := G/G_\Lambda$. We will need to know that the subsets $T(p)$ of T are not too pathological, in particular we want to know that each $T(p)$ admits an open neighbourhood of which $T(p)$ is a deformation retract.

– **4.7.** Let us recall some definitions and a theorem from real algebraic geometry. Let U be a nonempty open subset of some finite dimensional real vector space. A subset of U is said to be *semialgebraic* if it belongs to the smallest family of subsets of U which is closed under finite unions, finite intersections and complements, and which contains the sets

$$\{x \in U \mid f(x) \geq 0\}$$

for polynomial functions $f : U \rightarrow \mathbb{R}$. A central result in real algebraic geometry is that compact semialgebraic sets admit a finite triangulation. The following is a light version of Theorem 2.6.12 of [BR90]:

Theorem 4.8. *Let $X \subseteq \mathbb{R}^n$ be a compact semialgebraic set, and let X_1, \dots, X_m be a partition of X by semialgebraic subsets. There exists a finite simplicial complex (a polyhedron) K and a homeomorphism $h : X \rightarrow K$, such that each X_p is a union of sets $h^{-1}(F)$ for some relatively open faces F of K .*

The group G is itself a semialgebraic subset of \mathbb{R}^{r+s} , so we know what semialgebraic subsets of G are. But we wish to speak about semialgebraic sets on a real torus, say $T = G/G_\Lambda$. The real torus T can be given a structure of a compact locally semialgebraic space (which is: a locally ringed space,

locally isomorphic to a semialgebraic set, see [DK85], §I.1, Definition 3). Theorem 4.8 persists in this generality (*loc.cit.* §II.4, Theorem 4.4). In elementary terms however, this means the following: A subset X of T is called *semialgebraic* if there exist open subsets $U_i \subseteq \mathbb{R}^n$ which form an atlas of T via the projection maps $\varphi_i: U_i \rightarrow T$, such that each $\varphi_i^{-1}(X)$ is a semialgebraic subset of U_i .

Corollary 4.9. *Let T_1, \dots, T_m be a partition of T by semialgebraic subsets. There exists a finite simplicial complex K and a homeomorphism $h: T \rightarrow K$, such that each T_p is a union of sets $h^{-1}(F)$ for some relatively open faces F of K .*

We can deduce this corollary either from the triangulation theorem of [DK85], but also from the quoted Theorem 4.8. Indeed, choose for $X \subseteq G$ any compact semialgebraic set which contains a fundamental domain for T , and for $X_p \subseteq X$ the preimages in X of the subsets T_p . Possibly after subdividing some simplices, any triangulation $X \rightarrow K$ descends to a triangulation of T .

Lemma 4.10. *The set $T(1) \cup T(2) \cup \dots \cup T(p)$ is an open, semialgebraic subset of T .*

Proof. We start by showing that $G(1) \cup G(2) \cup \dots \cup G(p)$ is an open and locally semialgebraic subset of G . For every $g \in G$, let us denote by $E(g) \subseteq V$ the ellipsoid given by the equation

$$E(g) := \{v \in V \mid \|gv\| = \mu_n(g\Lambda)\}$$

and notice that $\mu_n(g\Lambda)$ is the largest real number r with the property that the lattice points which lie inside of $\{v \in V \mid \|gv\| = r\}$ do not generate V as a real vector space. The set of lattice points inside of $E(g)$ is indeed the set $D(g)$. Observe that $\mu_n(g\Lambda)$ is a continuous function of g , and that $E(g)$ varies continuously with g , say for the Hausdorff distance.

Pick an element $g_0 \in G(p)$. The set of those $g \in G$ with the property that every lattice point which is on the inside of $E(g_0)$ is also on the inside of $E(g)$, and every lattice point which is on the outside of $E(g_0)$ is also on the outside of $E(g)$, form an open neighbourhood of g_0 in G . This neighbourhood, call it U , is contained in

$$G(1) \cup G(2) \cup \dots \cup G(p) = \{g \in G \mid \dim M(g) \geq n - p\}$$

indeed, we have by definition $D(g_0) \subseteq D(g)$ and hence $M(g_0) \subseteq M(g)$ for all $g \in U$, and thus $U \subseteq G(1) \cup \dots \cup G(p)$. This shows that $G(1) \cup G(2) \cup \dots \cup G(p)$ is open.

Next, let us show that $G(q) \cup G(q+1) \cup \dots \cup G(p)$ is defined by polynomial inequalities on U for every $1 \leq q \leq p$. Let S_0 be the set of those lattice points which lie on $E(g_0)$. An element $g \in U$ belongs to $G(q) \cup \dots \cup G(p)$ if and only if the following conditions hold:

- (1) The set of points $D \subseteq S_0$ which lie inside of $E(g)$ satisfy $\dim \langle D, M(g_0) \rangle \leq n - q$.
- (2) The set of points $S \subseteq S_0$ which are on or inside $E(g)$ satisfy $\langle S, M(g_0) \rangle = V$.

The set of those $g \in U$ satisfying (1) and (2) for given sets $D \subseteq S$ is described by the quadratic polynomial inequalities $\|g\lambda_0\|^2 \geq \|g\lambda\|^2$ for $\lambda_0 \in S_0 \setminus D$ and $\lambda \in S \setminus D$, which in particular imply $\|g\lambda\| = \mu_n(g\Lambda)$ for $\lambda \in S \setminus D$. This shows that $(G(q) \cup \dots \cup G(p)) \cap U$ is a finite union of closed subsets of U , each of which is defined by finitely many polynomial inequalities, namely

$$(G(q) \cup \dots \cup G(p)) \cap U = \bigcup_{D \subseteq S} \{g \in U \mid \|g\lambda_0\|^2 \geq \|g\lambda\|^2 \text{ for all } \lambda_0 \in S_0 \setminus D, \lambda \in S \setminus D\}$$

where the union runs through all pairs of subsets $D \subseteq S$ of S_0 satisfying $\dim \langle D, M(g_0) \rangle \leq n - q$ and $\langle S, M(g_0) \rangle = V$.

The quotient map $G \rightarrow T$ is locally a polynomial diffeomorphism, that is a tautology. Hence $T(1) \cup T(2) \cup \dots \cup T(p)$ is open. Moreover, since T is compact, there exists a finite covering of T by open sets U with the property that $T(p+1) \cup \dots \cup T(n)$ is given on U as a finite union of closed subsets, each defined by finitely many polynomial inequalities. \square

Corollary 4.11. *Each subset $T(p) \subseteq T$ admits an open neighbourhood of which $T(p)$ is a deformation retract.*

Proof. By Corollary 4.9, there exists a finite simplicial complex K and a homeomorphism $h : T \rightarrow K$, such that each $T(p)$ is a union of sets $h^{-1}(F)$ for some relatively open faces F of K . But any union of relatively open faces on a simplicial complex, finite or not, admits an open neighbourhood which is a deformation retract, and we can transport these neighbourhoods to T via h . \square

Lemma 4.12. *Let $\gamma : [0, 1] \rightarrow T(p)$ be a path. Then $M(\gamma(0)) = M(\gamma(1))$.*

Proof. Pick $t_0 \in [0, 1]$ and set $g_0 := \gamma(t_0)$. We have seen in the proof of Lemma 4.10 that the inclusion $M(g_0) \subseteq M(g)$ holds for all g in some neighbourhood of g_0 , hence $M(\gamma(t_0)) \subseteq M(\gamma(t))$ holds for all t in some neighbourhood of t_0 . But we assume $\dim M(\gamma(t)) = n - p$ for all t , so the equality $M(\gamma(t_0)) = M(\gamma(t))$ must hold for all t close to t_0 . The map $t \mapsto M(\gamma(t))$ is therefore locally constant on $[0, 1]$, hence constant. \square

Lemma 4.13. *Let $M \subseteq V$ be a real linear subspace of dimension p which is generated by elements of Λ , and let $G_{M,\Lambda} \subseteq G_\Lambda$ be the subgroup consisting of those $g \in G_\Lambda$ satisfying $gM = M$. We have*

$$\text{rank}(G_{M,\Lambda}) < \text{g.c.d.}(n, p).$$

Proof. Consider the number field $k := \mathbb{Q}[G_{M,\Lambda}]$ and set $e := [k : \mathbb{Q}]$. We can regard $\Lambda \otimes \mathbb{Q}$ and $(M \cap \Lambda) \otimes \mathbb{Q}$ as k_0 -vector spaces, and thus have

$$e \mid \dim_{\mathbb{Q}}(\Lambda \otimes \mathbb{Q}) = n \quad \text{and} \quad e \mid \dim_{\mathbb{Q}}((M \cap \Lambda) \otimes \mathbb{Q}) = p$$

where the last equality holds since M is rational. Therefore, e divides $\text{g.c.d.}(n, p)$. Since $G_{M,\Lambda}$ embeds into the group of units \mathcal{O}_k^* , we have $\text{rank}(G_{M,\Lambda}) \leq \text{rank}(\mathcal{O}_k^*) \leq e - 1$, hence the claim. \square

Corollary 4.14. *For every $1 \leq p \leq n$ and every $t \in T(p)$, the image of the group homomorphism induced by the inclusion $T(p) \subseteq T = G/G_\Lambda$*

$$\rho : \pi_1(T(p), t) \rightarrow \pi_1(T, t) \cong G_\Lambda$$

has rank $\leq \text{g.c.d.}(n, p) - 1$.

Proof. Choose $g \in G(p)$ in the class of t . We show that the image of ρ is contained in $G_{M(g),\Lambda}$. Since $M(g)$ is generated by lattice elements and $\dim M(z) = n - p$ the desired conclusion follows then from Lemma 4.13. Let $[\gamma] \in \pi_1(T(p), t)$ be the class of a path $\gamma : [0, 1] \rightarrow T(p)$ such that $\gamma(0) = \gamma(1) = t$. Lift it to a path $\tilde{\gamma} : [0, 1] \rightarrow G$ with $\tilde{\gamma}(0) = g$. Setting $h := \rho([\gamma]) \in G_\Lambda$, we

have $h\tilde{\gamma}(0) = \tilde{\gamma}(1)$ by definition of the canonical isomorphism $\pi_1(T, t) \cong G_\Lambda$. By Lemma 4.12 the equality $M(\tilde{\gamma}(0)) = M(\tilde{\gamma}(1))$ holds, hence

$$M(g) = M(hg) = h^{-1}M(g),$$

and hence $h \in G_{M(g), \Lambda}$ as claimed. \square

Proof of Theorem 4.3. For $1 \leq p \leq n$, let U_p be an open neighborhood of $T(p)$ such that U_p is a retract of $T(p)$. In particular $T(p) = \emptyset$ if and only if $U_p = \emptyset$. Such neighborhoods exist by Corollary 4.11. We get an open covering $U_1 \cup \dots \cup U_n$ of T . Corollary 4.14 implies that

$$(12) \quad \text{rank}(H_1(W, \mathbb{Z}) \rightarrow H_1(T, \mathbb{Z})) < p$$

holds for every connected component W of U_p . It follows from (12) and theorem 4.1 that the sets U_1, \dots, U_{r+s-1} do not cover T , hence U_p and hence $T(p)$ must be nonempty for some $p \geq r+s$. This implies that there exists $g \in G$ with $\dim M(g) \leq s$, which was to be shown. \square

5. PROOF OF THE MAIN THEOREM

Equip the vector space $V = \mathbb{R}^r \oplus \mathbb{C}^s$ with the scalar product and the norm map introduced in 2.1, and let $G \subseteq \text{SL}_{r+s}(\mathbb{R})$ be the group of diagonal matrices with positive diagonal coefficients. Theorem 5.1 below gives an upper bound on the inhomogeneous minimum of every lattice $\Lambda \subseteq V$ whose G -orbit is compact. Our main Theorem stated in the introduction is a consequence of it.

Theorem 5.1. *Let Λ be a lattice in V such that $G\Lambda$ is compact. The following inequality holds for every $1 \leq a \leq r+s$.*

$$m(\Lambda)^s \cdot M(\Lambda)^a \leq (2^{s-a} \cdot \gamma_n^{s+a} \cdot n^{-s})^{\frac{n}{2}} \cdot \det(\Lambda)^{s+a}$$

Proof. By Theorem 4.3 there exists $g \in G$ such that $\mu_{s+1}(g\Lambda) = \dots = \mu_n(g\Lambda)$ holds. We have then

$$(13) \quad \mu_1(g\Lambda)^s \cdot \mu_n(g\Lambda)^a \leq \mu_1(g\Lambda) \cdot \dots \cdot \mu_s(g\Lambda) \cdot \mu_n(g\Lambda)^a \leq \gamma_n^{\frac{s+a}{2}} \cdot \det(g\Lambda)^{\frac{s+a}{n}}$$

by Lemma 3.3 with $t = s+a$. Furthermore, we have

$$(14) \quad \mu_n(g\Lambda) \geq \sqrt{2} \cdot M(g\Lambda)^{\frac{1}{n}} = \sqrt{2} \cdot M(\Lambda)^{\frac{1}{n}}$$

by Lemma 3.5, and

$$(15) \quad \mu_1(g\Lambda) \geq \frac{\sqrt{n}}{\sqrt{2}} \cdot m(g\Lambda)^{\frac{1}{n}} = \frac{\sqrt{n}}{\sqrt{2}} \cdot m(\Lambda)^{\frac{1}{n}}$$

by Lemma 3.4. The statement of the theorem follows by combining (13), (14) and (15). \square

– **5.2.** Let K be a number field of degree $n = r + 2s$ over \mathbb{Q} , with r real embeddings $\sigma_1, \dots, \sigma_r$ and s non-conjugated complex embeddings $\sigma_{r+1}, \dots, \sigma_{r+s}$. Let d_K be the absolute value of the discriminant of K . From the chosen ordering of the embeddings of K we obtain a \mathbb{Q} -linear map $\sigma : K \rightarrow V$ sending $x \in K$ to $(\sigma_1(x), \dots, \sigma_{r+s}(x))$. The image of \mathcal{O}_K under this map is a lattice $\Lambda := \sigma(\mathcal{O}_K) \subseteq V$ of volume $2^{-s} \cdot \sqrt{d_K}$ (see for example [Sam70], page 57).

Let $\epsilon : \mathcal{O}_K^* \rightarrow G$ be the group homomorphism sending $x \in K$ to the diagonal matrix with diagonal coefficients $|\sigma_1(x)|, \dots, |\sigma_{r+s}(x)|$. Indeed we have $\det(\epsilon(x)) = N(x) = 1$. The kernel of ϵ is the finite group of roots of unity in \mathcal{O}_K^* . The maps σ and ϵ are compatible, in the sense that

$$\sigma(xy) = \epsilon(x)\sigma(y)$$

holds for all $x \in \mathcal{O}_K^*$ and $y \in K$. In particular we have $\epsilon(\mathcal{O}_K^*) \subseteq G_\Lambda$, hence $G_\Lambda \subseteq G$ has rank $r + s - 1$ by Dirichlet's Unit Theorem. It follows that $G\Lambda \cong G/G_\Lambda$ is compact.

Theorem 5.3 (Main Theorem). *Let K be a number field of signature (r, s) and degree $n = r + 2s$, and let d_K be the absolute value of the discriminant of K . Then*

$$M(\Lambda) \leq 2^{-\frac{s(s+a)}{a}} \cdot (2^{s-a} \cdot \gamma_n^{s+a} \cdot n^{-s})^{\frac{n}{2a}} \cdot d_K^{\frac{s+a}{2a}}$$

holds for every $1 \leq a \leq r + s$.

Proof. As we have explained in 5.2, the lattice $\Lambda := \sigma(\mathcal{O}_K) \subseteq V$ associated with K has a compact G -orbit. Since we have $m(\Lambda) = 1$, the inequality follows directly from Theorem 5.1. \square

– **5.4.** For number fields with a large discriminant, the choice $a = r + s$ will give the best upper bound in Theorem 5.3, whereas for number fields with a small discriminant also other choices for a can be interesting. For $a = r + s$ and using the estimate $\gamma_n \leq \frac{n}{2}$ for $n \geq 4$ (Theorem 2.7.4. of [Mar03] and page 17 of [MH73]) we obtain the theorem stated in the introduction.

Remark 5.5. For small degrees n , the exact value of the Hermite constant γ_n is known. The following table presents the explicit bounds that we obtain from Theorem 5.3 for $n \leq 5$. To the authors knowledge, these bounds are the best known for $n = 4, 5$ and $s \neq 0$.

n	s	Upper bound for $M(K)$
1	0	$\frac{1}{\sqrt{2}} \cdot \sqrt{d_K}$
2	0	$\frac{1}{\sqrt{3}} \cdot \sqrt{d_K}$
2	1	$\frac{1}{6} \cdot d_K$
3	0	$\frac{1}{2} \cdot \sqrt{d_K}$
3	1	$\min\left(\frac{1}{6\sqrt{3}} \cdot d_K, \frac{1}{2\sqrt[3]{108}} \cdot d_K^{\frac{3}{4}}\right)$
4	0	$\frac{1}{2} \cdot \sqrt{d_K}$
4	1	$\min\left(\frac{1}{16} \cdot d_K, \frac{1}{8} \cdot d_K^{\frac{3}{4}}, \frac{1}{4\sqrt[3]{4}} \cdot d_K^{\frac{2}{3}}\right)$
4	2	$\min\left(\frac{1}{512} \cdot d_K^{\frac{3}{2}}, \frac{1}{64} \cdot d_K\right)$
5	0	$\frac{1}{2} \cdot \sqrt{d_K}$
5	1	$\min\left(\frac{2}{25\sqrt{5}} \cdot d_K, \frac{1}{4\sqrt[4]{20}} \cdot d_K^{\frac{3}{4}}, \frac{1}{2\sqrt[6]{3125}} \cdot d_K^{\frac{2}{3}}, \frac{1}{2\sqrt[8]{12500}} \cdot d_K^{\frac{5}{8}}\right)$
5	2	$\min\left(\frac{2}{3125} \cdot d_K^{\frac{3}{2}}, \frac{1}{50\sqrt{5}} \cdot d_K, \frac{1}{10\sqrt[3]{100}} \cdot d_K^{\frac{5}{6}}\right)$

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